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The nonholonomic variational principle

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Abstract

A variational principle for mechanical systems and fields subject to nonholonomic constraints is found, providing Chetaev-reduced equations as equations for extremals. Investigating nonholonomic variations of the Chetaev type and their properties, we develop foundations of the calculus of variations on constraint manifolds, modelled as fibred submanifolds in jet bundles. This setting is appropriate to study general first-order 'nonlinear nonitegrable constraints' that locally are given by a system of first-order ordinary or partial differential equations. We obtain an invariant constrained first variation formula and constrained Euler–Lagrange equations both in intrinsic and coordinate forms, and show that the equations are the same as Chetaev equations 'without Lagrange multipliers', introduced recently by other methods. We pay attention to two possible settings: first, when the constrained system arises from an unconstrained Lagrangian system defined in a neighbourhood of the constraint, and second, more generally, when an 'internal' constrained system on the constraint manifold is given. In the latter case a corresponding unconstrained system need not be a Lagrangian, nor even exist. We also study in detail an important particular case: nonholonomic constraints that can be alternatively modelled by means of (co)distributions in the total space of the fibred manifold; in nonholonomic mechanics this happens whenever constraints affine in velocities are considered. It becomes clear that (and why) if the distribution is completely integrable (= the constraints are semiholonomic), the principle of virtual displacements holds and can be used to obtain the constrained first variational formula by a more or less standard procedure, traditionally used when unconstrained or holonomic systems are concerned. If, however, the constraint is nonintegrable, no significant simplifications are available. Among others, some properties of nonholonomic systems are clarified that without a deeper insight seem rather mysterious.

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1. Introduction

This paper is concerned with the problem of existence of a *variational principle providing nonholonomic equations of motion as equations for extremals*. Similarly as in classical mechanics of systems subject to holonomic constraints, motion equations of a Lagrangian system in the presence of constraints depending on velocities can be investigated from a 'mechanical' and a 'geometrical' point of view. The former approach reflects the *physical* understanding of constrained dynamics as *motions in the original configuration space subject to reactive forces* expressing the constraints. Mathematically this leads to *equations of motion with Lagrange multipliers*. In nonholonomic mechanics they take the form as follows, first conjectured by Chetaev [4]:

$$\frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q^\sigma} = \mu_a \frac{\partial f^a}{\partial \dot q^\sigma}, \qquad 1 \leqslant \sigma \leqslant m, \tag{1.1}$$

where m is the dimension of the configuration space,

$$f^a(t, q^\sigma, \dot q^\sigma) = 0, \qquad 1 \leqslant a \leqslant k < m, \tag{1.2}$$

are equations of constraints, and μ_a are Lagrange multipliers. Chetaev equations have to be solved simultaneously with equations of the constraints (1.2), providing in this way a system of $m + k$ mixed first- and second-order ODEs for $m + k$ functions: constrained curves $c(t) = (q^\sigma(t))$ and Lagrange multipliers μ_a. Various geometric settings for Chetaev equations are subject of many papers (e.g. [3, 6–10, 14, 23, 25, 26]), a recent generalization to field theory is also available [1, 16, 30, 29].

We adopt the latter viewpoint reflecting a *geometrical* understanding of constrained dynamics as *motions on the constraint manifold*. Mathematically the dynamics are described as solutions of a *reduced system of equations* where the unknown reaction forces are absent (equations 'without Lagrange multipliers', equivalent with Chetaev equations). A geometric setting providing reduced equations is due to [24] (Lagrangian systems subject to nonholonomic constraints in mechanics), [14] (general mechanical systems with nonholonomic constraints) and [16, 19, 20] (field theory). In a frequent situation of a first-order mechanical Lagrangian system subject to first-order nonholonomic constraints, reduced equations read as follows:

$$\frac{\partial_c \bar L}{\partial q^s} - \frac{\mathrm{d}_c}{\mathrm{d}t}\frac{\partial \bar L}{\partial \dot q^s} - L_a\left(\frac{\partial_c g^{m-k+a}}{\partial q^s} - \frac{\mathrm{d}_c}{\mathrm{d}t}\frac{\partial g^{m-k+a}}{\partial \dot q^s}\right) = 0, \qquad 1 \leqslant s \leqslant m-k, \tag{1.3}$$

where the functions g^{m-k+a} are defined by

$$\dot q^{m-k+a} = g^{m-k+a}(t, q^\sigma, \dot q^1, \ldots, \dot q^{m-k}), \qquad 1 \leqslant a \leqslant k, \tag{1.4}$$

i.e., (1.4) are equations of the constraints (1.2) in normal form, $\bar L$ is the Lagrangian L restricted to the constraint submanifold, L_a is a shorthand notation for

$$L_a = \frac{\partial L}{\partial \dot q^{m-k+a}}(t, q^\sigma, \dot q^1, \ldots, \dot q^{m-k}, g^{m-k+1}, \ldots, g^m), \tag{1.5}$$

and the 'constraint derivative' operators read

$$\begin{aligned} \frac{\partial_c}{\partial q^s} &= \frac{\partial}{\partial q^s} + \frac{\partial g^{m-k+a}}{\partial \dot q^s}\frac{\partial}{\partial q^{m-k+a}}, \\ \frac{\mathrm{d}_c}{\mathrm{d}t} &= \frac{\partial}{\partial t} + \sum_{s=1}^{m-k}\dot q^s\frac{\partial}{\partial q^s} + \sum_{a=1}^{k} g^{m-k+a}\frac{\partial}{\partial q^{m-k+a}} + \sum_{s=1}^{m-k}\ddot q^s\frac{\partial}{\partial \dot q^s}. \end{aligned} \tag{1.6}$$

There is a natural question if nonholonomic equations (1.3) can be obtained from a variational principle as corresponding equations for extremals (Euler–Lagrange equations). It is well known that the standard variational principle does not apply to this situation. The application of a variational procedure to nonholonomic linear and nonlinear constraints is troublesome in many points (cf [2, 9, 22, 27]). Analysing the problem, it turns out that answers to the following questions have to be found.

- How to generalize the principle of virtual displacements? We need a geometrically satisfactory concept of 'nonholonomic virtual displacements' and reactive forces compatible with the constraints.
- What are variations compatible with a nonholonomic constraint?
- What is a constrained Lagrangian system? Surprisingly, a 'Lagrange function' defined on the constraint is not sufficient to obtain nonholonomic equations of motion. Moreover, as found in [18], where a concept of variationality for nonholonomic systems has been proposed, based on a relation between the Euler–Lagrange operator and the exterior derivative, a non-Lagrangian system may become variational if subject to an appropriate nonholonomic constraint.

In the present paper, we answer these questions and obtain a variational principle for mechanical systems and fields subject to nonholonomic constraints. We stress that our point of view is to consider *dynamical systems given on a constraint manifold, and governed by reduced equations as equations of motion*. The results that might be rather surprising appear through a careful analysis of basic concepts (such as variations, virtual displacements, Lagrangians, etc) and the geometry of nonholonomic constraints.

In section 2 we recall from [11] how the invariant first variation formula on fibred manifolds is obtained, suitable for description of unconstrained and holonomic Lagrangian systems in mechanics and field theory. Keeping this procedure in mind, we can better understand differences appearing due to the presence of nonholonomic constraints.

Solution of the nonholonomic variations problem needs a deeper understanding of first-order differential constraints and related geometric structures. We study differential forms and vector fields on nonholonomic submanifolds in section 3, and prove a 'constraint version' of the theorem on decomposition of differential forms into contact components. We also generalize the prolongation of vector fields to the constrained situation. Techniques we develop are used in the next section to obtain and prove the results on constrained variations. The key to a correct concept of nonholonomic virtual displacements, or admissible variations, is the *canonical distribution* [14, 16, 23]; we recall it in section 3.3.

In the last section we develop foundations of the calculus of variations on constraint manifolds. Our setting is appropriate to study general first-order 'nonlinear nonitegrable constraints' that locally are given by a system of first-order ordinary or partial differential equations. We obtain an invariant constrained first variation formula and the corresponding equations for extremals ('constrained Euler–Lagrange equations') both in intrinsic and coordinate forms, and show that, indeed, the latter equations are the same as Chetaev-reduced equations 'without Lagrange multipliers'.

It is important to stress that we pay attention to two possible settings: first, when the constrained system arises from an unconstrained Lagrangian system defined in a neighbourhood of the constraint, and second, more general, when an 'internal' constrained system on the constraint manifold is given. In the latter case a corresponding unconstrained system need not be Lagrangian, or even need not exist. (In this context, we recall the interested reader that necessary and sufficient conditions on 'constraint variationality', generalizing the famous Helmholtz conditions, have been obtained in [18]).

We also study in detail an important particular case: nonholonomic constraints that can be alternatively modelled by means of (co)distributions in the total space of the fibred manifold. In nonholonomic mechanics this happens whenever constraints are affine in velocities [14], in field theory, however, the situation is not so simple [16]. It becomes clear that (*and why*) if the distribution is completely integrable, i.e., the constraints are *semiholonomic*, the principle of virtual displacements holds and can be used to obtain the constrained first variational formula by a more or less standard procedure, traditionally used when unconstrained or holonomic systems are concerned. If, however, the constraint is *nonintegrable*, we shall see that remarkable simplifications are no longer available, and the 'general approach' has to be applied.

In the last section we illustrate the nonholonomic variational principle on a few examples, and discuss in detail main differences compared with a traditional approach and expectations based on experience with unconstrained and holonomic systems.

To give insight into the nonholonomic variation procedure for the moment, let us consider an easy example of a free particle moving in $\mathbb{R}^3$ under a nonintegrable constraint on velocity. The unconstrained motion can be described by sections of the fibred manifold $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, (i.e., by graphs of curves in $\mathbb{R}^3$), and comes from the Lagrangian 1-form $\lambda = L\,\mathrm{d}t$, defined on $\mathbb{R} \times T\mathbb{R}^3$, where

$$L = \tfrac{1}{2}mv^2. \tag{1.7}$$

Variations of curves in $\mathbb{R}^3$ are generated by vector fields

$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z}, \tag{1.8}$$

and induce variations of prolonged curves in the evolution space $T\mathbb{R}^3$, generated by prolongations of the variation vector fields in $\mathbb{R}^3$, providing the Euler–Lagrange equations

$$m\ddot{x} = 0, \qquad m\ddot{y} = 0, \qquad m\ddot{z} = 0. \tag{1.9}$$

It is important to note that the same result is obtained if in the action function the Cartan form

$$\Theta_\lambda = -\tfrac{1}{2}mv^2\,\mathrm{d}t + m\dot{x}\,\mathrm{d}x + m\dot{y}\,\mathrm{d}y + m\dot{z}\,\mathrm{d}z, \tag{1.10}$$

in place of the Lagrangian $L\,\mathrm{d}t$ is considered. Assume a nonholonomic constraint given by equation

$$v^2 = t, \qquad t > 0, \tag{1.11}$$

or, in a normal form,

$$\dot{z} = g \equiv \sqrt{t - \dot{x}^2 - \dot{y}^2}. \tag{1.12}$$

This equation defines a submanifold $\iota : Q \to \mathbb{R} \times T\mathbb{R}^3$ of codimension one in $\mathbb{R} \times T\mathbb{R}^3$. In Q the constrained dynamics take place; hence Q has the meaning of a genuine evolution space for the constrained system. The manifold Q carries the canonical distribution that determines admissible 'virtual displacements'; it is annihilated by 1-form

$$-\frac{t}{g}\,\mathrm{d}t + \frac{\dot{x}}{g}\,\mathrm{d}x + \frac{\dot{y}}{g}\,\mathrm{d}y + \mathrm{d}z. \tag{1.13}$$

The constrained variation principle will concern variations of graphs of curves in this submanifold. We shall show that the constrained action function comes from the 1-form

$$\iota^*\Theta_\lambda = -\tfrac{1}{2}mt\,\mathrm{d}t + m\dot{x}\,\mathrm{d}x + m\dot{y}\,\mathrm{d}y + mg\,\mathrm{d}z, \tag{1.14}$$

and that the constrained variations are generated by vector fields

$$\frac{\partial}{\partial x} - \frac{\dot{x}}{g}\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial y} - \frac{\dot{y}}{g}\frac{\partial}{\partial z} \tag{1.15}$$

(tangent to Q). Note that these variations do not come from variations in $\mathbb{R} \times \mathbb{R}^3$–a typical property of nonholonomic constraints. As a result we then obtain constrained Euler–Lagrange equations that in this case take the following form:

$$m\ddot{x}\left(1+\frac{\dot{x}^2}{g^2}\right)+\frac{m\dot{x}\dot{y}}{g^2}\ddot{y}-\frac{m\dot{x}}{2g^2}=0, \qquad m\ddot{y}\left(1+\frac{\dot{y}^2}{g^2}\right)+\frac{m\dot{x}\dot{y}}{g^2}\ddot{x}-\frac{m\dot{y}}{2g^2}=0. \tag{1.16}$$

2. A reminder of the first variation formula on fibred manifolds

In what follows, we consider smooth manifolds and mappings. In coordinate formulae, summation over repeated indices applies.

Let $\pi : Y \to X$ be a fibred manifold with an orientable base $X, \dim X = n \geqslant 1, \dim Y = n + m$ $(m \geqslant 1)$. We denote by $\pi_r : J^rY \to X$ the r-jet prolongation of π, and $\pi_{r,s} : J^rY \to J^sY, r > s \geqslant 0$, the canonical jet projections. For the sake of simplicity of notations we also consider $r = 0$ and write $Y = J^0Y, \pi = \pi_0$, etc. In this paper, we mostly use the first and second jet prolongations, J^1Y and J^2Y.

If $\dim X = 1$, local fibred coordinates on Y and the associated coordinates on J^rY are denoted by (t, q^σ), and $\left(t, q^\sigma, q^\sigma_1, \ldots, q^\sigma_r\right)$, respectively; if, in particular $r \leqslant 2$, we write $(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$. In the case of $\dim X > 1$ we use notation (x^i, y^σ), and $\left(x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1\ldots j_r}\right)$, where $1 \leqslant \sigma \leqslant m, 1 \leqslant i, j_1, \ldots, j_r \leqslant n, j_1 \leqslant \cdots \leqslant j_r$.

A mapping $\delta : X \to J^rY$ defined on an open set $U \subset X$ is called a *section* of π_r if $\pi_r \circ \delta = \mathrm{id}_U$. Sections of π can be prolonged to sections of $\pi_r, r \geqslant 1$. Recall that if γ is a section of π, in fibred coordinates $\gamma(t) = (t, \gamma^\sigma(t))$ (resp. $\gamma(x^i) = (x^i, \gamma(x^i))$), then $J^r\gamma$ is a section of π_r, $J^r\gamma(t) = (t, \gamma^\sigma(t), \mathrm{d}\gamma^\sigma/\mathrm{d}t, \ldots, \mathrm{d}^r\gamma^\sigma/\mathrm{d}t^r)$ (resp. $J^r\gamma(x^i) = (x^i, \gamma^\sigma(x^i), \partial\gamma^\sigma/\partial x^i, \ldots, \partial^r\gamma^\sigma/\partial x^{i_1}\cdots\partial x^{i_r})$).

A section δ of π_r is called *holonomic* if $\delta = J^r\gamma$ for a section γ of π.

A form η on J^rY is called *contact* if $J^r\gamma^*\eta = 0$ for every section γ of π. η is called *horizontal* or *0-contact* if $i_\xi\eta = 0$ for every π_r-vertical vector field ξ on J^rY. For $1 \leqslant k \leqslant q$, a contact q-form η is called *k-contact* if for every π_r-vertical vector field ξ on $J^rY, i_\xi\eta$ is $(k-1)$-contact. By *Krupka's decomposition theorem*, every q-form η on J^rY is canonically decomposed into a sum of uniquely determined q-forms on $J^{r+1}Y$, a horizontal form $h\eta$, 1-contact form $p_1\eta, \ldots, q$-contact form $p_q\eta$, called the *horizontal*, *1-contact*, ..., *q-contact component* of η, respectively [11]. Hence,

$$\pi^*_{r+1,r}\eta = h\eta + p_1\eta + \cdots + p_q\eta. \tag{2.1}$$

For a function, f, this formula gives us

$$\pi^*_{r+1,r}\,\mathrm{d}f = h\,\mathrm{d}f + p_1\,\mathrm{d}f, \tag{2.2}$$

where $h\,\mathrm{d}$ and $p_1\,\mathrm{d}$ are the *horizontal derivative* and the *contact derivative* operator, respectively. In fibred coordinates, components of $h\,\mathrm{d}$ are the well-known *total derivative* operators

$$\begin{aligned} &\text{if } \dim X = 1: && \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \sum_{k=0}^{r} q^\sigma_{k+1}\frac{\partial}{\partial q^\sigma_k}, \\ &\text{if } \dim X > 1: && \frac{\mathrm{d}}{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} + \sum_{k=0}^{r} y^\sigma_{j_1\cdots j_k i}\frac{\partial}{\partial y^\sigma_{j_1\cdots j_k}}, \qquad 1 \leqslant i \leqslant n. \end{aligned} \tag{2.3}$$

Components of p_1 d are partial derivatives, $\partial/\partial q^\sigma_k, 0 \leqslant k \leqslant r$, resp. $\partial/\partial y^\sigma_{j_1\ldots j_k}, 0 \leqslant k \leqslant r$.

Vector fields on Y that are π-projectable (i.e., their projection is a vector field on X) admit prolongations to vector fields on J^rY, $r \geqslant 1$. Recall that, in fibred coordinates, if

$$\xi = \xi^0 \frac{\partial}{\partial t} + \xi^\sigma \frac{\partial}{\partial q^\sigma}, \qquad \text{resp.} \quad \xi = \xi^j \frac{\partial}{\partial x^j} + \xi^\sigma \frac{\partial}{\partial y^\sigma}, \tag{2.4}$$

(where $\xi^0 = \xi^0(t)$, resp. $\xi^j = \xi^j(x^i)$) then

$$J^r\xi = \xi + \sum_{k=1}^{r} \xi_k^\sigma \frac{\partial}{\partial q_k^\sigma}, \qquad \text{resp.} \quad J^r\xi = \xi + \sum_{k=1}^{r} \xi^\sigma_{j_1\cdots j_k} \frac{\partial}{\partial y^\sigma_{j_1\cdots j_k}}, \tag{2.5}$$

where the higher components take the form

$$\xi_k^\sigma = \frac{\mathrm{d}\xi^\sigma_{k-1}}{\mathrm{d}t} - q_k^\sigma \frac{\mathrm{d}\xi^0}{\mathrm{d}t}, \qquad \text{resp.} \quad \xi^\sigma_{j_1\cdots j_{k-1}i} = \frac{\mathrm{d}\xi^\sigma_{j_1\cdots j_{k-1}}}{\mathrm{d}x^i} - y^\sigma_{j_1\cdots j_{k-1}l} \frac{\partial \xi^l}{\partial x^i}. \tag{2.6}$$

We note that prolonged vector fields are *symmetries of the contact ideal*.

Throughout the paper we use the following shorthand notation:

$$\omega_0 = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \qquad \omega_i = i_{\partial/\partial x^i}\omega_0,$$

and

$$\begin{aligned} &\text{if } \dim X = 1: \quad \omega^\sigma = \mathrm{d}q^\sigma - \dot q^\sigma \mathrm{d}t, \qquad && \dot\omega^\sigma = \mathrm{d}\dot q^\sigma - \ddot q^\sigma\, \mathrm{d}t, \\ &\text{if } \dim X > 1: \quad \omega^\sigma = \mathrm{d}y^\sigma - y^\sigma_j\, \mathrm{d}x^j, \qquad && \omega^\sigma_i = \mathrm{d}y^\sigma_i - y^\sigma_{ij}\, \mathrm{d}x^j. \end{aligned}$$

A *dynamical form* E of order 2 is defined to be a 1-contact form on J^2Y, horizontal with respect to the projection onto Y. In fibred coordinates, $E = E_\sigma\omega^\sigma \wedge \omega_0$, where E_σ, $1 \leqslant \sigma \leqslant m$, are functions on an open subset of J^2Y. Second-order dynamical forms represent systems of second-order differential equations, ordinary if $\dim X = 1$ and partial if $\dim X = n > 1$, for sections of the fibred manifold π.

Let us briefly recall some concepts and results on the first-order variational calculus on fibred manifolds, due to Krupka [11] (see also [12] or [13]).

A first-order *Lagrangian* is a *horizontal* n-form λ on J^1Y. In fibred coordinates $\lambda = L\omega_0$, where L is a function on an open subset of J^1Y. A *Lepage equivalent* of λ is an n-form ρ such that $h\rho = \lambda$ and $p_1\, \mathrm{d}\rho$ is a *dynamical form*. If $\dim X = 1$ then λ has a *unique* Lepage equivalent, the well-known *Cartan form*,

$$\Theta_\lambda = L\, \mathrm{d}t + \frac{\partial L}{\partial \dot q^\sigma}\omega^\sigma. \tag{2.7}$$

For $\dim X > 1$ a Lepage equivalent is no longer unique; the family of Lepage equivalents of λ takes the form

$$\rho = \Theta_\lambda + \mathrm{d}\nu + \mu, \tag{2.8}$$

where

$$\Theta_\lambda = L\omega_0 + \frac{\partial L}{\partial y^\sigma_j}\omega^\sigma \wedge \omega_j \tag{2.9}$$

is the Poincaré–Cartan form, ν is an arbitrary contact $(n-1)$-form and μ is an arbitrary at least 2-contact form. It has to be stressed, however, that the $(n+1)$-form $p_1\, \mathrm{d}\rho$ depends only upon the horizontal part λ of ρ, i.e. is *the same* for all Lepage equivalents of the Lagrangian λ; it is called the *Euler–Lagrange* form of λ, and denoted by E_λ.

The procedure providing the 'unconstrained' invariant *first variation formula* and *Euler–Lagrange equations* on fibred manifolds can be summarized as follows.

Let Ω be a piece of X (i.e., a compact connected n-dimensional submanifold of X with boundary). Denote by $\mathcal{S}_\Omega(\pi)$ the set of sections of π, domains of which are neighbourhoods of Ω. Given a Lagrangian λ on J^1Y, the function

$$S_{\lambda,\Omega} : \mathcal{S}_\Omega(\pi) \ni \gamma \to \int_\Omega J^1\gamma^*\lambda \in \mathbb{R} \tag{2.10}$$

is called the *action function* of λ over Ω.

Note that since we are interested in critical paths *in Y*, the action of *any n-form* η, such that $h\eta = \lambda$, coincides with the action of the Lagrangian λ. In particular, over every piece Ω of X, *the action of* λ *is the same as the action of any of the Lepage equivalents* ρ *of* λ,

$$S_{\lambda,\Omega} = S_{\rho,\Omega}. \tag{2.11}$$

To get a correct concept of *variation* (1-parametric deformation) of a section $\gamma \in \mathcal{S}_\Omega(\pi)$, one has to consider π*-projectable* vector fields on Y. The point is that such vector fields transfer *sections into sections*: if ξ is a projectable vector field on Y and ξ_0 (on X) is its projection, and $\{\phi_u\}$, resp. $\{\phi_{0u}\}$ are the corresponding local 1-parameter groups, we get a 1-parameter family of *sections*, $\gamma_u = \phi_u\gamma\phi_{0u}^{-1}$, defined in a neighbourhood of $\phi_{0u}(\Omega)$, and called *variation of the section* γ *induced by* ξ. Thus, for a fixed section γ and a fixed 'variation vector field' ξ we get a real function

$$u \to \int_{\phi_{0u}(\Omega)} J^1\gamma_u^*\lambda. \tag{2.12}$$

The arising function

$$\delta S_{\lambda,\Omega} : \mathcal{S}_\Omega(\pi) \ni \gamma \to \left(\frac{\mathrm{d}}{\mathrm{d}u}\int_{\phi_{0u}(\Omega)} J^1\gamma_u^*\lambda\right)_{u=0} = \int_\Omega J^1\gamma^*\mathcal{L}_{J^1\xi}\lambda \in \mathbb{R} \tag{2.13}$$

is called the *first variation of the action function* of the Lagrangian λ over Ω, *induced by* ξ, or the *first variational derivative of* $S_{\lambda,\Omega}$ *by* ξ. It should be stressed that since the operator $\mathcal{L}_{J^1\xi}$ preserves the decomposition of forms into the horizontal and contact components, we have also

$$\delta S_{\lambda,\Omega} = \delta S_{\rho,\Omega} \tag{2.14}$$

for every Lepage equivalent ρ *of* λ. Explicitly,

$$\int_\Omega J^1\gamma^*\mathcal{L}_{J^1\xi}\lambda = \int_\Omega J^1\gamma^*\mathcal{L}_{J^1\xi}\rho. \tag{2.15}$$

The first variation formula is a decomposition of the above integral into a sum of two terms such that the first one does not depend upon 'derivations of variations' (the Euler–Lagrange term) and the second one is a boundary term. With *Lepage forms* the decomposition is available in an *invariant* (geometric) way simply by using Cartan's formula for the Lie derivative of ρ. The first variation formula then can be stated either in the *integral form* as

$$\begin{aligned}\int_\Omega J^1\gamma^*\mathcal{L}_{J^1\xi}\lambda &= \int_\Omega J^1\gamma^* i_{J^1\xi}\,\mathrm{d}\rho + \int_\Omega J^1\gamma^*\mathrm{d}i_{J^1\xi}\rho \\ &= \int_\Omega J^1\gamma^* i_{J^1\xi}\,\mathrm{d}\rho + \int_{\partial\Omega} J^1\gamma^* i_{J^1\xi}\rho,\end{aligned} \tag{2.16}$$

or in the *infinitesimal form*

$$\mathcal{L}_{J^1\xi}\lambda = h i_{J^1\xi}\,\mathrm{d}\rho + h\,\mathrm{d}i_{J^1\xi}\rho. \tag{2.17}$$

Note that due to properties of Lepage forms (up to a projection) $hi_{J^1\xi}\,\mathrm{d}\rho = hi_{J^2\xi}\,p_1\,\mathrm{d}\rho = hi_{J^2\xi}E_\lambda$, so that indeed, the first term on the right-hand side of the first variation formula is the Euler–Lagrange term, and we may equivalently write

$$\int_\Omega J^1\gamma^*\mathcal{L}_{J^1\xi}\lambda = \int_\Omega J^2\gamma^* i_{J^2\xi}E_\lambda + \int_{\partial\Omega} J^1\gamma^* i_{J^1\xi}\rho, \tag{2.18}$$

or

$$\mathcal{L}_{J^1\xi}\lambda = hi_{J^2\xi}E_\lambda + h\,\mathrm{d}i_{J^1\xi}\rho. \tag{2.19}$$

From the above form of the first variation formula it is immediately seen that *the first variation formula does not depend upon a choice of the Lepage equivalent* ρ *of* λ, i.e. it reads

$$\mathcal{L}_{J^1\xi}\lambda = hi_{J^2\xi}E_\lambda + h\,\mathrm{d}i_{J^1\xi}\Theta_\lambda. \tag{2.20}$$

Consequently, also *the 'boundary term' providing conserved currents is completely determined by the Lagrangian*. This can easily be seen also by a direct computation as follows: since $\rho = \Theta_\lambda + \mathrm{d}\nu + \mu$, where ν is contact and μ is at least 2-contact, we get

$$h\,\mathrm{d}i_{J^1\xi}\rho = h\,\mathrm{d}i_{J^1\xi}\Theta_\lambda + h\,\mathrm{d}i_{J^1\xi}\,\mathrm{d}\nu, \tag{2.21}$$

since $\mathrm{d}i_{J^1\xi}\mu$ is contact as the exterior derivative of a contact form. Moreover,

$$h\,\mathrm{d}i_{J^1\xi}\,\mathrm{d}\nu = h\mathcal{L}_{J^1\xi}\,\mathrm{d}\nu = 0, \tag{2.22}$$

since $\mathrm{d}\nu$ is contact and $\mathcal{L}_{J^1\xi}$ preserves contact forms.

A section γ of π is called an *extremal of* λ *on* Ω if the first variation of the action of λ on Ω vanishes for every *vertical* vector field ξ on Y with the support in $\pi^{-1}(\Omega)$ (such a vector field is often called a *fixed-endpoints variation*). γ is called *extremal of* λ if it is an extremal on every piece $\Omega \subset X$.

Equations for extremals of a Lagrangian are called *Euler–Lagrange equations*. Using the first variation formula, it can be proved that they are as follows [11].

Theorem 2.1. *Let* λ *be a Lagrangian on* J^1Y. *A section* γ *of* π *is an extremal of* λ*, if and only if* γ *satisfies one of the following equivalent conditions.*

(1) $E_\lambda \circ J^2\gamma = 0$.
(2) For every vertical vector field ξ *on* Y, $J^1\gamma^* i_{J^1\xi}\,\mathrm{d}\rho = 0$, *where* ρ *is (any) Lepage equivalent of* λ.
(3) For every projectable vector field ξ *on* Y, $J^1\gamma^* i_{J^1\xi}\,\mathrm{d}\rho = 0$, *where* ρ *is (any) Lepage equivalent of* λ.
(4) For every vector field ζ *on* J^1Y, $J^1\gamma^* i_\zeta \mathrm{d}\rho = 0$, *where* ρ *is (any) Lepage equivalent of* λ.
(5) In every fibred chart (x^i, y^σ) *on* Y, γ *satisfies the system of differential equations*

$$\frac{\partial L}{\partial y^\sigma} - \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial L}{\partial y^\sigma_j} = 0, \qquad 1 \leqslant \sigma \leqslant m. \tag{2.23}$$

Remark 2.2. Note that the above variational principle applies to *unconstrained systems* and to systems with *holonomic constraints* (indeed, geometrically, a holonomic constraint in Y is a fibred submanifold $\pi|_Q : Q \to X$ of the fibred manifold $\pi : Y \to X$).

Before turning to a possible generalization to nonholonomic systems, let us stress two remarkable points appearing in the variation procedure:

- *Virtual displacements.* The geometric setting gives a justification and precise formulation to an 'obvious fact' that *'derivations of variations are variations of derivations'*, observed and used in classical mechanics of unconstrained systems and systems with holonomic constraints. Indeed, in the above geometric language this statement means nothing but the elementary property that for any variation $\{\gamma_u\}$ of a section γ of $\pi : Y \to X$

$$J^1(\gamma_u) = (J^1\gamma)_u \tag{2.24}$$

for all admissible values of the parameter u.
Another formulation of the same property can be given in terms of 'virtual displacements' appearing in D'Alembert's principle: since infinitesimal virtual displacements are represented by vertical vector fields on Y (resp. on the holonomic constraint submanifold $Q \subset Y$), induced virtual displacements in J^1Y (resp. in J^1Q) are *prolongations* of these vector fields.

- *Lepage forms.* For a Lagrangian system, the action is

$$\mathcal{S}_\Omega(\pi) \ni \gamma \to \int_\Omega J^1\gamma^*\rho \in \mathbb{R}, \tag{2.25}$$

where ρ is a *Lepage form*. The pleasant fact that ρ can be replaced by its horizontal part (Lagrangian λ) is a favourable feature of the prolongation structure of the manifold J^1Y (resp. J^1Q).

3. Nonholonomic constraints

3.1. Constraint submanifolds in jet bundles

By a *nonholonomic constraint* in J^1Y we mean a submanifold $Q \subset J^1Y$, fibred over Y. This means that we have the fibred manifolds $\bar\pi_{1,0} : Q \to Y$ where $\bar\pi_{1,0}$ is the restriction of the projection $\pi_{1,0} : J^1Y \to Y$ to Q, and $\bar\pi_1 : Q \to X$, where $\bar\pi_1 = \pi_1|_Q$. We also use the explicit notation $\iota : Q \to J^1Y$ for the canonical embedding and write

$$\dot q^\sigma \circ \iota = g^\sigma, \qquad \text{respectively,} \quad y^\sigma_j \circ \iota = g^\sigma_j. \tag{3.1}$$

By definition of Q,

$$\operatorname{rank}\left(\frac{\partial g^\sigma}{\partial \dot q^\nu}\right) = m - k, \qquad \operatorname{rank}\left(\frac{\partial g^\sigma_j}{\partial y^\nu_k}\right) = nm - \kappa, \tag{3.2}$$

where k, resp. κ is the codimension of Q (we exclude the cases $k = m$, $\kappa = nm$ when Q is the image of a section of $J^1Y \to Y$—trivial fibres).

The contact ideal on J^1Y gives rise on Q to the *induced contact ideal*, consisting of pullbacks by ι to Q of contact forms on J^1Y. The contact ideal on Q is generated by 1-forms

$$\bar\omega^\sigma = \iota^*\omega^\sigma, \qquad 1 \leqslant \sigma \leqslant m, \tag{3.3}$$

and their exterior derivatives.

If $\dim X = 1$, then a constraint $Q \subset J^1Y$ of codimension k ($1 \leqslant k < m$) is locally defined by *a system of k first-order ordinary differential equations*

$$f^a(t, q^\sigma, \dot q^\sigma) = 0, \qquad 1 \leqslant a \leqslant k, \tag{3.4}$$

where the functions f^a satisfy the rank condition

$$\operatorname{rank}\left(\frac{\partial f^a}{\partial \dot q^\sigma}\right) = k, \tag{3.5}$$

or, equivalently, by equations 'in normal form'

$$\dot{q}^{m-k+a} = g^{m-k+a}(t, q^\sigma, \dot{q}^1, \ldots, \dot{q}^{m-k}), \qquad 1 \leqslant a \leqslant k. \tag{3.6}$$

Hence, the embedding ι is explicitly given by equations

$$\dot{q}^s \circ \iota = \dot{q}^s, \qquad 1 \leqslant s \leqslant m-k, \qquad \dot{q}^{m-k+a} \circ \iota = g^{m-k+a}, \qquad 1 \leqslant a \leqslant k. \tag{3.7}$$

On the submanifold Q we have adapted coordinates $(t, q^\sigma, \dot{q}^s)$, where $1 \leqslant s \leqslant m-k$.

If $\dim X = n > 1$, then a constraint $Q \subset J^1Y$ of codimension κ $(1 \leqslant \kappa < nm)$ is locally defined by *a system of* κ *first-order partial differential equations*

$$f^\alpha(x^i, y^\sigma, y^\sigma_j) = 0, \qquad 1 \leqslant \alpha \leqslant \kappa, \tag{3.8}$$

where the functions f^α satisfy the rank condition

$$\operatorname{rank}\left(\frac{\partial f^\alpha}{\partial y^\sigma_j}\right) = \kappa, \tag{3.9}$$

where $1 \leqslant \alpha \leqslant \kappa$ number rows and σ, j, $1 \leqslant \sigma \leqslant m$, $1 \leqslant j \leqslant n$ number columns.

Adapted coordinates on Q are denoted by (x^i, y^σ, z^J), $1 \leqslant J \leqslant mn - \kappa$. Here z^J stands for the coordinates y^p_l for appropriate p's from the set $\{1, 2, \ldots, m\}$ and l's from the set $\{1, 2, \ldots, n\}$; let us denote this set of pairs of admissible indices by $\mathcal{J}$. Note that due to the rank condition (3.9) equations $f^\alpha = 0$ are locally equivalent to κ equations of the form $y^\sigma_j = g^\sigma_j(x^i, y^\nu, z^J)$ for $(\sigma, j) \notin \mathcal{J}$ (cf (3.2)).

Constraints can be naturally prolonged to higher order jets. The *first prolongation* $\hat{Q}$ of the constraint Q is a submanifold in J^2Y, consisting of all points $J^2_x\gamma$ such that $J^1_x\gamma \in Q$, $x \in X$. Locally $\hat{Q}$ is defined by the equations of the constraint and their derivatives, precisely,

$$f^a = 0, \qquad \frac{\mathrm{d}f^a}{\mathrm{d}t} = 0, \qquad 1 \leqslant a \leqslant k, \tag{3.10}$$

respectively, in normal form,

$$\dot{q}^{m-k+a} = g^{m-k+a}, \qquad \ddot{q}^{m-k+a} = \frac{\mathrm{d}g^{m-k+a}}{\mathrm{d}t}, \tag{3.11}$$

if $\dim X = 1$, and

$$f^\alpha = 0, \qquad \frac{\mathrm{d}f^\alpha}{\mathrm{d}x^i} = 0, \qquad 1 \leqslant \alpha \leqslant \kappa, \quad 1 \leqslant i \leqslant n, \tag{3.12}$$

respectively,

$$y^\sigma_j = g^\sigma_j, \qquad y^\sigma_{jl} = \frac{\mathrm{d}g^\sigma_j}{\mathrm{d}x^l}, \qquad (\sigma, j) \notin \mathcal{J}, \quad 1 \leqslant l \leqslant n \tag{3.13}$$

if $\dim X = n$. Note that generally $y^\sigma_{jl} \neq y^\sigma_{lj}$.

We also use notation $\hat{\iota} : \hat{Q} \to J^2Y$ for the canonical embedding. The manifold $\hat{Q}$ is fibred over Q, Y and X, the fibred projections are simply restrictions of the corresponding canonical projections of the underlying fibred manifolds. We write $\bar{\pi}_2 : \hat{Q} \to X$, $\bar{\pi}_{2,1} : \hat{Q} \to Q$ and $\bar{\pi}_{2,0} : \hat{Q} \to Y$. On $\hat{Q}$ we use adapted fibred coordinates, denoted by $(t, q^\sigma, \dot{q}^s, \ddot{q}^s)$, $1 \leqslant \sigma \leqslant m$, $1 \leqslant s \leqslant m-k$ if $\dim X = 1$ and by $\left(x^i, y^\sigma, z^J, z^J_j\right)$ if $\dim X = n$.

In what follows, whenever using coordinates on Q or $\hat{Q}$, we mean *adapted coordinates* of this kind.

On $\hat{Q}$ there arises the induced contact ideal generated by the 1-forms

$$\bar{\omega}^s = \mathrm{d}q^s - \dot{q}^s\,\mathrm{d}t, \qquad \bar{\omega}^{m-k+a} = \mathrm{d}q^{m-k+a} - g^{m-k+a}\,\mathrm{d}t, \qquad \hat{\omega}^s = \mathrm{d}\dot{q}^s - \ddot{q}^s\,\mathrm{d}t, \tag{3.14}$$

respectively,

$$\bar{\omega}^\sigma = \mathrm{d}y^\sigma - g^\sigma_j\,\mathrm{d}x^j, \qquad \hat{\omega}^J = \mathrm{d}z^J - z^J_j\,\mathrm{d}x^j, \tag{3.15}$$

and their exterior derivatives.

We have a natural concept of a *contact symmetry* on Q, resp. $\hat{Q}$, as a vector field on Q, resp. $\hat{Q}$, that is a symmetry of the induced contact ideal.

3.2. Vector fields and differential forms on constraint submanifolds

The contact structure on the constraint and its prolongations enable us to consider *constrained prolongations of vector fields* as follows: let ξ be a vector field on Y. We call a vector field ζ on Q the *first constrained prolongation* of ξ, and denote it by $J^1_c\xi$, if ζ is a contact symmetry on Q and $T\bar{\pi}_{1,0}\cdot\zeta = \xi\circ\bar{\pi}_{1,0}$. Similarly, we call $\hat{\zeta}$ on $\hat{Q}$ the *second constrained prolongation* of ξ, and denote it by $J^2_c\xi$, if $\hat{\zeta}$ is a contact symmetry on $\hat{Q}$ and $T\bar{\pi}_{2,0}\cdot\hat{\zeta} = \xi\circ\bar{\pi}_{2,0}$.

Also modules of differential forms on constraint manifolds inherit an additional structure due to the existence of the fibred and contact structure.

A q-form η on Q is called *horizontal* with respect to the projection $\bar{\pi}_1$ if $i_\zeta\eta = 0$ for every $\bar{\pi}_1$-vertical vector field ζ on Q. Quite similarly we define horizontality with respect to the projection $\bar{\pi}_{1,0}$ onto Y, and several concepts of horizontality for forms on $\hat{Q}$ (note that one has forms, horizontal with respect to the projection $\bar{\pi}_2$ onto X, $\bar{\pi}_{2,0}$ onto Y and finally $\bar{\pi}_{2,1}$ onto Q).

By recurrence, a contact q-form η on Q (respectively, on $\hat{Q}$) is called *i-contact*, $i = 1, 2, \ldots, q$, if for every $\bar{\pi}_1$-vertical vector field ζ on Q (respectively, $\bar{\pi}_2$-vertical vector field ζ on $\hat{Q}$) the contraction of η by ζ is $(i-1)$-contact (here 0-contact means horizontal with respect to the projection onto X).

We have structure theorems, similar to the decomposition theorem in the unconstrained case:

Theorem 3.1. *Let $q \geqslant 1$.*

(i) Denote by $\Lambda^q_Q(\hat{Q})$ the module of q-forms on $\hat{Q}$ that are horizontal with respect to the projection $\bar{\pi}_{2,1}$ onto Q, and by $\Lambda^{q-i,i}_Q(\hat{Q})$ its submodules of $\bar{\pi}_2$-horizontal ($i = 0$) and i-contact forms ($i = 1, 2, \ldots, q$). Then

$$\Lambda^q_Q(\hat{Q}) = \Lambda^{q,0}_Q(\hat{Q}) \oplus \Lambda^{q-1,1}_Q(\hat{Q}) \oplus \cdots \oplus \Lambda^{0,q}_Q(\hat{Q}). \tag{3.16}$$

This means that every form $\hat{\eta} \in \Lambda^q_Q(\hat{Q})$ is in a unique way decomposed into the sum of a horizontal form and i-contact forms, $i = 1, 2, \ldots q$.

(ii) With analogous notations as above,

$$\Lambda^q_Y(Q) = \Lambda^{q,0}_Y(Q) \oplus \Lambda^{q-1,1}_Y(Q) \oplus \cdots \oplus \Lambda^{0,q}_Y(Q). \tag{3.17}$$

In view of the above theorem we can consider for every $q \geqslant 1$ the projectors of the module $\Lambda^q_Q(\hat{Q})$ onto the particular submodules in the decomposition (3.16). We denote $\bar{h} : \Lambda^q_Q(\hat{Q}) \to \Lambda^{q,0}_Q(\hat{Q})$, and $\bar{p}_i : \Lambda^q_Q(\hat{Q}) \to \Lambda^{q-i,i}_Q(\hat{Q})$, $1 \leqslant i \leqslant q$, and speak about the *horizontal* and *i-contact component* of a form $\hat{\eta} \in \Lambda^q_Q(\hat{Q})$.

Since, in particular, for every q-form η on Q its lift $\bar{\pi}^*_{2,1}\eta$ belongs to $\Lambda^q_Q(\hat{Q})$, we obtain the following corollary.

Corollary 3.2. *For every q-form η on Q one has a unique decomposition into a sum of a $\bar{\pi}_2$-horizontal form and i-contact forms, $i = 1, 2, \ldots q$, on $\hat{Q}$ as follows:*

$$\bar{\pi}^*_{2,1}\eta = \bar{h}\eta + \bar{p}_1\eta + \cdots + \bar{p}_q\eta. \tag{3.18}$$

Proof. The proof of the theorem is based on transformation properties of horizontal and contact forms on $\hat{Q}$. It is convenient to work in the basis of 1-forms on $\hat{Q}$, adapted to the induced contact structure, that is, $\left(dx^i, \bar{\omega}^\sigma, \hat{\omega}^J, dz^J_j\right)$. The condition $\hat{\eta} \in \Lambda^q_Q(\hat{Q})$ means that the coordinate expression of $\hat{\eta}$ does not contain the differentials dz^J_j. Hence, in a chart, $\hat{\eta}$ is expressed as a sum of the following q-forms: $\hat{\eta}_0$ containing wedge products of the dx^i's only, $\hat{\eta}_1$ that is a sum of forms containing the wedge product of $q - 1\,\mathrm{d}x^i$'s and exactly one of the contact 1-forms $\bar{\omega}^\sigma$, $\hat{\omega}^J$, $\hat{\eta}_2$ being a sum of terms that contain the wedge product with exactly two of the forms $\bar{\omega}^\sigma$, $\hat{\omega}^J$, etc, up to $\hat{\eta}_q$ that contains the wedge products of q of the $\bar{\omega}^\sigma$, $\hat{\omega}^J$'s. Obviously, $\hat{\eta}_0$ is a horizontal form, $\hat{\eta}_1$ is 1-contact, etc. The decomposition $\hat{\eta} = \hat{\eta}_0 + \hat{\eta}_1 + \cdots + \hat{\eta}_q$ is, however, invariant with respect to fibred coordinate transformations (the number of $\bar{\omega}^\sigma$, $\hat{\omega}^J$'s in a wedge product of $\mathrm{d}x^i$'s, $\bar{\omega}^\sigma$'s and $\hat{\omega}^J$'s does not change). □

Applying the above corollary to (locally) exact 1-forms on Q gives an invariant *splitting of the exterior derivative* d to the horizontal and contact part, $\bar{\pi}^*_{2,1}\,\mathrm{d} = \bar{h}\,\mathrm{d} + \bar{p}_1\mathrm{d}$. The operator $\bar{h}\,\mathrm{d}$ has $n = \dim X$ components as follows,

$$\frac{\mathrm{d_c}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}^s\frac{\partial}{\partial q^s} + g^{m-k+a}\frac{\partial}{\partial q^{m-k+a}} + \ddot{q}^s\frac{\partial}{\partial \dot{q}^s}, \tag{3.19}$$

respectively,

$$\frac{\mathrm{d_c}}{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} + g_i^\sigma\frac{\partial}{\partial y^\sigma} + z_i^J\frac{\partial}{\partial z^J}, \tag{3.20}$$

and is called the *constraint total derivative*.

For convenience of notations we also put

$$\frac{\mathrm{d'_c}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}^s\frac{\partial}{\partial q^s} + g^{m-k+a}\frac{\partial}{\partial q^{m-k+a}}, \qquad \frac{\mathrm{d'_c}}{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} + g_i^\sigma\frac{\partial}{\partial y^\sigma}. \tag{3.21}$$

With the help of notations introduced above it is easy to write explicit formulae for a vector field $J^1_{\mathrm{c}}\xi$. For $\dim X = 1$ we obtain the following result (if $\dim X > 1$, the formulae are more complicated, however, the computation is straightforward):

Theorem 3.3. *Let* $\dim X = 1$. *A projectable vector field* ξ *on* Y,

$$\xi = \xi^0\frac{\partial}{\partial t} + \xi^\sigma\frac{\partial}{\partial q^\sigma}, \tag{3.22}$$

admits prolongation to Q *if and only if*

$$\begin{aligned}\frac{\mathrm{d'_c}\xi^{m-k+a}}{\mathrm{d}t} - \frac{\partial g^{m-k+a}}{\partial q^{m-k+b}}\xi^{m-k+b} &= \frac{\partial g^{m-k+a}}{\partial t}\xi^0 + \frac{\partial g^{m-k+a}}{\partial q^l}\xi^l \\ &\quad + \frac{\partial g^{m-k+a}}{\partial \dot{q}^l}\frac{\mathrm{d'_c}\xi^l}{\mathrm{d}t} + \left(g^{m-k+a} - \frac{\partial g^{m-k+a}}{\partial \dot{q}^l}\dot{q}^l\right)\frac{\mathrm{d'_c}\xi^0}{\mathrm{d}t}.\end{aligned} \tag{3.23}$$

Then

$$J^1_{\mathrm{c}}\xi = \xi + \left(\frac{\mathrm{d'_c}\xi^l}{\mathrm{d}t} - \dot{q}^l\frac{\mathrm{d'_c}\xi^0}{\mathrm{d}t}\right)\frac{\partial}{\partial \dot{q}^l}. \tag{3.24}$$

It is obvious how to consider *higher order* prolongations of a constraint $Q \subset J^1Y$, and how to treat horizontal and contact forms in the higher order situation. In this paper, however, higher order constraint structures will not be needed.

3.3. The canonical distribution

As discovered in [14] and [23] for mechanics and in [16] for field theory, every nonholonomic constraint $Q \subset J^1Y$ carries a natural structure, called the *canonical distribution*, denoted by $\mathcal{C}$. It should be stressed that this structure gives a *geometric meaning to 'virtual displacements' in* *the space of positions and velocities, and to the concept of 'reactive forces'* (see [14] and [16] for a *nonholonomic D'Alembert's principle* and introduction and study of Chetaev forces).

Mechanics. If $\dim X = 1$ then the canonical distribution is a *corank k distribution on Q*, where $k = \operatorname{codim} Q$, annihilated by the following system of k linearly independent smooth *contact 1-forms*:

$$\varphi^a = \left(\frac{\partial f^a}{\partial \dot{q}^\sigma} \circ \iota\right) \bar{\omega}^\sigma = \bar{\omega}^{m-k+a} - \sum_{s=1}^{m-k} \frac{\partial g^{m-k+a}}{\partial \dot{q}^s} \bar{\omega}^s, \qquad 1 \leqslant a \leqslant k. \tag{3.25}$$

Note that the condition $\operatorname{rank}\mathcal{C} = \text{constant}$ means that $\mathcal{C} \to Q$ is a *bundle* over Q (a subbundle of the tangent bundle $TQ \to Q$); it is also called the *Chetaev bundle*.

The ideal in the exterior algebra on Q generated by the 1-forms φ^a, $1 \leqslant a \leqslant k$, is called the *constraint ideal*, and is denoted by $\mathcal{I}(\mathcal{C}^0)$. Differential forms belonging to the constraint ideal are called *constraint forms*.

Equivalently, the canonical distribution can locally be spanned by a system of $2(m-k)+1$ smooth *vector fields* on Q:

$$\begin{aligned}
&\frac{\partial_c}{\partial t} \equiv \frac{\partial}{\partial t} + \sum_{a=1}^{k} \left(g^{m-k+a} - \sum_{l=1}^{m-k} \frac{\partial g^{m-k+a}}{\partial \dot{q}^l} \dot{q}^l \right) \frac{\partial}{\partial q^{m-k+a}} \\
&\frac{\partial_c}{\partial q^s} \equiv \frac{\partial}{\partial q^s} + \sum_{a=1}^{k} \frac{\partial g^{m-k+a}}{\partial \dot{q}^s} \frac{\partial}{\partial q^{m-k+a}}, \qquad 1 \leqslant s \leqslant m-k, \\
&\frac{\partial}{\partial \dot{q}^s}, \qquad 1 \leqslant s \leqslant m-k.
\end{aligned} \tag{3.26}$$

In what follows, we shall call vector fields belonging to the canonical distribution *Chetaev vector fields*. Note that every Chetaev vector field takes the form

$$\begin{aligned}
Z &= Z^0 \frac{\partial_c}{\partial t} + \sum_{s=1}^{m-k} Z^s \frac{\partial_c}{\partial q^s} + \sum_{s=1}^{m-k} \tilde{Z}^s \frac{\partial}{\partial \dot{q}^s} \\
&= Z^0 \frac{\partial}{\partial t} + \sum_{s=1}^{m-k} Z^s \frac{\partial}{\partial q^s} \\
&\quad + \sum_{a=1}^{k} \left(Z^0 g^{m-k+a} + \sum_{s=1}^{m-k} (Z^s - Z^0 \dot{q}^s) \frac{\partial g^{m-k+a}}{\partial \dot{q}^s} \right) \frac{\partial}{\partial q^{m-k+a}} + \sum_{s=1}^{m-k} \tilde{Z}^s \frac{\partial}{\partial \dot{q}^s},
\end{aligned} \tag{3.27}$$

where the components $Z^0, Z^s, \tilde{Z}^s$ of Z are functions of the variables $(t, q^\sigma, \dot{q}^s)$, $1 \leqslant \sigma \leqslant m$, $1 \leqslant s \leqslant m-k$, on Q.

It is immediately seen that *the family of Chetaev vector fields need not contain vector fields projectable onto Y*. Moreover, the canonical distribution $\mathcal{C}$ need not contain *prolongations of vector fields defined on Y*, even if it is projectable.

Remarkably, the following theorem holds, first observed and proved in [14].

Theorem 3.4. *The constraint Q is given by equations affine in the first derivatives if and only if the canonical distribution on Q is $\bar{\pi}_{1,0}$-projectable (i.e., the projection $\mathcal{D}$ of $\mathcal{C}$ is a distribution on Y).*

Note that in the affine case, if we denote $g^{m-k+a} = A^a + B^a_s \dot{q}^s$, the distribution $\mathcal{D}$ is locally spanned by vector fields

$$\frac{\partial}{\partial t} + \sum_{a=1}^{k} A^a \frac{\partial}{\partial q^{m-k+a}}, \qquad \frac{\partial}{\partial q^s} + \sum_{a=1}^{k} B^a_s \frac{\partial}{\partial q^{m-k+a}}, \quad 1 \leqslant s \leqslant m-k, \tag{3.28}$$

or, annihilated by 1-forms $A^a\, \mathrm{d}t + B^a_s\, \mathrm{d}q^s - \mathrm{d}q^{m-k+a}$, $1 \leqslant a \leqslant k$.

An important particular case concerns *semiholonomic constraints*, properties of which can be summarized as follows [14, 15].

Theorem 3.5. *Given a nonholonomic constraint $Q \subset J^1Y$, the following conditions are equivalent:*

(1) Q is semiholonomic.

(2) The canonical distribution $\mathcal{C}$ on Q is completely integrable.

(3) $\mathcal{C}$ on Q is projectable onto Y, and its projection $\mathcal{D} = T\bar{\pi}_{1,0}.\mathcal{C}$ is completely integrable.

(4) The constraint ideal $\mathcal{I}(\mathcal{C}^0)$ is closed.

(5) Functions g^{m-k+a} defining locally the constraint satisfy

$$\mathcal{E}^{\mathrm{c}}(g^{m-k+a}) = 0, \qquad 1 \leqslant a \leqslant k, \tag{3.29}$$

where $\mathcal{E}^{\mathrm{c}}$ denotes the $\mathcal{C}$-modified Euler–Lagrange operator, in components defined as follows:

$$\mathcal{E}^{\mathrm{c}}_s = \frac{\partial_c}{\partial q^s} - \frac{\mathrm{d_c}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}^s}, \qquad 1 \leqslant s \leqslant m-k, \tag{3.30}$$

where

$$\frac{\mathrm{d_c}}{\mathrm{d}t} = \frac{\partial_\mathrm{c}}{\partial t} + \dot{q}^s \frac{\partial_\mathrm{c}}{\partial q^s} + \ddot{q}^s \frac{\partial}{\partial \dot{q}^s}. \tag{3.31}$$

(6) Functions g^{m-k+a} defining locally the constraint are affine in velocities and satisfy

$$\mathcal{E}'^{\mathrm{c}}(g^{m-k+a}) = 0, \qquad 1 \leqslant a \leqslant k, \tag{3.32}$$

where

$$\mathcal{E}'^{\mathrm{c}}_s = \frac{\partial_c}{\partial q^s} - \frac{\mathrm{d}'_\mathrm{c}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}^s}, \qquad 1 \leqslant s \leqslant m-k. \tag{3.33}$$

Above, $\mathrm{d}'_\mathrm{c}/\mathrm{d}t = \mathrm{d_c}/\mathrm{d}t - \ddot{q}^s \partial_\mathrm{c}/\partial \dot{q}^s$.

An easy computation with the help of theorem 3.3 gives us that for semiholonomic constraints, the projections of vector fields $\partial_\mathrm{c}/\partial t$ and $\partial_\mathrm{c}/\partial q^l$, $1 \leqslant l \leqslant m-k$, *admit the prolongation to Q*. This means that vector fields $J^1(T\bar{\pi}_{1,0} \cdot \partial_\mathrm{c}/\partial t)$ and $J^1(T\bar{\pi}_{1,0} \cdot \partial_\mathrm{c}/\partial q^l)$ are along Q tangent to Q, and belong to the distribution $\mathcal{C}$. Indeed, computing the prolongation condition of theorem 3.3 we obtain in the case of $\partial_\mathrm{c}/\partial q^l$ and $\partial_\mathrm{c}/\partial t$, respectively:

$$\frac{\mathrm{d}'_\mathrm{c}}{\mathrm{d}t}\frac{\partial g^{m-k+a}}{\partial \dot{q}^l} - \frac{\partial g^{m-k+a}}{\partial q^{m-k+b}}\frac{\partial g^{m-k+b}}{\partial \dot{q}^l} - \frac{\partial g^{m-k+a}}{\partial q^l} = \frac{\mathrm{d}'_\mathrm{c}}{\mathrm{d}t}\frac{\partial g^{m-k+a}}{\partial \dot{q}^l} - \frac{\partial_\mathrm{c} g^{m-k+a}}{\partial q^l} = 0,$$

$$\begin{aligned}
\frac{\mathrm{d}'_\mathrm{c}}{\mathrm{d}t}&\left(g^{m-k+a} - \frac{\partial g^{m-k+a}}{\partial \dot{q}^l}\dot{q}^l\right) - \frac{\partial g^{m-k+a}}{\partial q^{m-k+b}}\left(g^{m-k+b} - \frac{\partial g^{m-k+b}}{\partial \dot{q}^l}\dot{q}^l\right) - \frac{\partial g^{m-k+a}}{\partial t} \\
&= \frac{\partial g^{m-k+a}}{\partial q^l}\dot{q}^l - \frac{\mathrm{d}'_\mathrm{c}}{\mathrm{d}t}\left(\frac{\partial g^{m-k+a}}{\partial \dot{q}^l}\dot{q}^l\right) + \frac{\partial g^{m-k+a}}{\partial q^{m-k+b}}\frac{\partial g^{m-k+b}}{\partial \dot{q}^l}\dot{q}^l \\
&= \left(\frac{\partial_c g^{m-k+a}}{\partial q^l} - \frac{\mathrm{d}'_\mathrm{c}}{\mathrm{d}t}\frac{\partial g^{m-k+a}}{\partial \dot{q}^l}\right)\dot{q}^l = 0,
\end{aligned} \tag{3.34}$$

in view of the above theorem. Summarizing, we have the following theorem.

Theorem 3.6. *The canonical distribution $\mathcal{C}$ of a semiholonomic constraint is spanned by vector fields $J^1_c\xi$, where ξ belongs to the projection $\mathcal{D}$ of $\mathcal{C}$, and $\bar{\pi}_{1,0}$-vertical vector fields.*

Remark 3.7. Note that by the above theorems, *constraints linear or affine in velocities can be alternatively modelled by means of a distribution on Y*, and similarly, *semiholonomic constraints can be modelled by means of a completely integrable distribution on Y*. The geometric description of nonholonomic constraints by a distribution on Y (on a 'configuration space', or 'space of events') is quite popular and frequently used. The reader should, however, keep in mind that using such a model in mechanics means that *exactly* constraints *affine in velocities* are considered, while *in field theory this is no longer true* [16].

To better understand the structure of the constraint ideal, it is worth noting that

$$d\varphi^a = \psi^a + \text{2-contact form} + \text{ constraint form}, \tag{3.35}$$

where (with the above notations)

$$\psi^a = -\mathcal{E}'^{c}_s(g^{m-k+a})\bar{\omega}^s \wedge dt + \frac{\partial^2 g^{m-k+a}}{\partial\dot{q}^r\partial\dot{q}^s}\bar{\omega}^r \wedge d\dot{q}^s, \tag{3.36}$$

hence,

$$\bar{p}_1\psi^a = -\mathcal{E}^{c}_s(g^{m-k+a})\bar{\omega}^s \wedge dt. \tag{3.37}$$

Field theory. If $\dim X > 1$, the situation is more complicated (see [16]). The canonical distribution $\mathcal{C}$ on Q is annihilated by the following system of κn *smooth contact 1-forms*

$$\phi^{\alpha j} = \left(\frac{\partial f^\alpha}{\partial y^\sigma_j} \circ \iota\right)\bar{\omega}^\sigma, \qquad 1 \leqslant \alpha \leqslant \kappa, \quad 1 \leqslant j \leqslant n, \tag{3.38}$$

that, however, are not linearly independent. Moreover, the rank of $\mathcal{C}$ need not be constant ($\mathcal{C} \to Q$ need not be a *bundle* over Q).

We can see that at each point in Q,

$$\operatorname{corank}\mathcal{C} = \operatorname{rank}\left(\frac{\partial f^\alpha}{\partial y^\sigma_j} \circ \iota\right) \leqslant \min\{m, \kappa n\}, \tag{3.39}$$

where the right-hand side matrix has κn rows labelled by α, j, and m columns, labelled by σ. We say that the constraint Q is *regular* if the matrix (3.39) has a constant rank, k, where $1 \leqslant k < m$. We then call k the *constraint dimension* of Q.

If the constraint Q is regular, then $\mathcal{C}$ is locally annihilated by a system of k linearly independent 1-forms of (3.38), and we may assume generators of $\mathcal{C}$ in the following *normal form*

$$\varphi^a = \bar{\omega}^{m-k+a} - \sum_{s=1}^{m-k} G^a_s\bar{\omega}^s, \qquad 1 \leqslant a \leqslant k, \tag{3.40}$$

where G^a_s are appropriate functions.

Similarly as above, by *constraint forms* we shall mean differential forms belonging to the constraint ideal on Q, generated by the 1-forms φ^a, $1 \leqslant a \leqslant k$.

The canonical distribution of a *regular* constraint can locally be spanned by a system of smooth vector fields

$$\frac{\partial_c}{\partial x^i} = \frac{\partial}{\partial x^i} + \sum_{a=1}^{k}\left(g^{m-k+a}_i - G^a_s g^s_i\right)\frac{\partial}{\partial y^{m-k+a}}, \qquad 1 \leqslant i \leqslant n,$$

$$\frac{\partial_c}{\partial y^s} = \frac{\partial}{\partial y^s} + \sum_{a=1}^{k} G^a_s \frac{\partial}{\partial y^{m-k+a}}, \qquad 1 \leqslant s \leqslant m-k, \tag{3.41}$$
$$\frac{\partial}{\partial z^J}, \qquad 1 \leqslant J \leqslant nm-\kappa,$$

hence *Chetaev vector fields* become vector fields on Q of the following form:

$$\begin{aligned} Z &= Z^i_0 \frac{\partial_c}{\partial x^i} + \sum_{s=1}^{m-k} Z^s \frac{\partial_c}{\partial y^s} + \sum_{J=1}^{nm-\kappa} \tilde{Z}^J \frac{\partial}{\partial z^J} \\ &= Z^i_0 \frac{\partial}{\partial x^i} + \sum_{s=1}^{m-k} Z^s \frac{\partial}{\partial y^s} + \sum_{a=1}^{k} \left(Z^i_0 g^{m-k+a}_i + \sum_{s=1}^{m-k} G^a_s \left(Z^s - Z^i_0 g^s_i\right)\right) \frac{\partial}{\partial y^{m-k+a}} + \sum_{J=1}^{nm-\kappa} \tilde{Z}^J \frac{\partial}{\partial z^J}. \end{aligned} \tag{3.42}$$

One can see that $\mathcal{C}$ can be spanned by vector fields *projectable onto X*.

Similarly as in mechanics, *the family of Chetaev vector fields need not contain vector fields projectable onto Y*, so that *the canonical distribution $\mathcal{C}$ need not contain prolongations of vector fields defined on Y*. We have the following result (see [16]).

Theorem 3.8. *The following conditions are equivalent.*

(1) $\mathcal{C}$ is $\bar{\pi}_{1,0}$-projectable (i.e., the projection $\mathcal{D}$ of $\mathcal{C}$ is a distribution on Y).

(2) The relation
$$\frac{\partial G^a_s}{\partial z^J} = 0 \tag{3.43}$$
holds.

(3) The constraint Q is locally given by equations
$$f^\alpha \equiv f^a_i = A^a_i + B^a_\sigma y^\sigma_i = 0, \qquad \operatorname{rank}\left(B^a_\sigma\right) = k, \tag{3.44}$$
where $1 \leqslant \alpha = (a, i) \leqslant \kappa = \operatorname{codim} Q$, $1 \leqslant a \leqslant k$, $1 \leqslant i \leqslant n$.

(4) The constraint Q is defined by a distribution on Y, locally annihilated by 1-forms
$$A^a_i \,\mathrm{d}x^i + B^a_\sigma \mathrm{d}y^\sigma, \qquad \operatorname{rank}\left(B^a_\sigma\right) = k, \tag{3.45}$$
where $1 \leqslant a \leqslant k$.

Note that equations (3.44) represent only a *particular form* of PDE's affine in the first derivatives. Thus in field theory, constraints modelled by a (co)distribution on Y represent only a 'small family' of affine constraints: for example, the constraint $y^1_1 + y^2_2 = 0$ does not come from a distribution on Y. This makes nonholonomic field theory distinct from mechanics, where *all* constraints affine in the first derivatives can be modelled by a distribution on Y.

A constraint Q is called *semiholonomic* if the constraint ideal is closed, i.e., the canonical distribution $\mathcal{C}$ is completely integrable [16]. It can be shown that the canonical distribution on every semiholonomic constraint is $\bar{\pi}_{1,0}$*-projectable* [16]. Hence, *every semiholonomic constraint can be alternatively* modelled as a weakly horizontal[1] completely integrable distribution of corank *k on Y*. Moreover, in complete analogy with mechanics, semiholonomic constraints in field theory agree with the prolongation structure in J^1Y. Analysing the canonical distribution in the same way, we get the same result as in theorem 3.6: *the canonical distribution $\mathcal{C}$ on a semiholonomic constraint is spanned by vector fields of the form $J^1_c\xi$ where ξ belongs to the projection $\mathcal{D}$ of $\mathcal{C}$, and $\bar{\pi}_{1,0}$-vertical vector fields.*

[1] The 'weak horizontality' is expressed by condition $\operatorname{rank}(B^a_\sigma) = k$ and means that sections of π are among admissible integral mappings.

The following theorem will be very useful to simplify many formulae and calculations[2].

Theorem 3.9. *Every regular nonholonomic constraint $Q \subset J^1Y$ has the following properties.*

(1) The forms $\bar{p}_1\, \mathrm{d}\varphi^a$, $1 \leqslant k \leqslant a$, are $\bar{\pi}_{1,0}$-horizontal.

(2) For every constraint 1-form φ, $\bar{p}_1\, \mathrm{d}\varphi$ is $\bar{\pi}_{1,0}$-horizontal.

(3) The following identities are true:

$$G^a_s \left(\frac{\mathrm{d}_c g^s_j}{\mathrm{d}x^i} - \frac{\mathrm{d}_c g^s_i}{\mathrm{d}x^j} \right) = \frac{\mathrm{d}_c g^{m-k+a}_j}{\mathrm{d}x^i} - \frac{\mathrm{d}_c g^{m-k+a}_i}{\mathrm{d}x^j}, \tag{3.46}$$

$$G^a_s \left(\frac{\mathrm{d}'_c g^s_j}{\mathrm{d}x^i} - \frac{\mathrm{d}'_c g^s_i}{\mathrm{d}x^j} \right) = \frac{\mathrm{d}'_c g^{m-k+a}_j}{\mathrm{d}x^i} - \frac{\mathrm{d}'_c g^{m-k+a}_i}{\mathrm{d}x^j}, \tag{3.47}$$

$$\mathcal{C}^a_{Jj} \equiv G^a_s \frac{\partial g^s_j}{\partial z^J} - \frac{\partial g^{m-k+a}_j}{\partial z^J} = 0. \tag{3.48}$$

Proof. Computing $\mathrm{d}\varphi^a$ we obtain

$$\begin{aligned}
\mathrm{d}\varphi^a &= \mathrm{d}\bar{\omega}^{m-k+a} - \mathrm{d}G^a_s \wedge \bar{\omega}^s - G^a_s\, \mathrm{d}\bar{\omega}^s \\
&= -\mathrm{d}g^{m-k+a}_j \wedge \mathrm{d}x^j - \mathrm{d}G^a_s \wedge \bar{\omega}^s + G^a_s\, \mathrm{d}g^s_j \wedge \mathrm{d}x^j \\
&= \left(G^a_s \frac{\mathrm{d}'_c g^s_j}{\mathrm{d}x^i} - \frac{\mathrm{d}'_c g^{m-k+a}_j}{\mathrm{d}x^i} \right) \mathrm{d}x^i \wedge \mathrm{d}x^j \\
&\quad + \left(G^a_s \frac{\partial_c g^s_j}{\partial y^r} + \frac{\mathrm{d}'_c G^a_r}{\mathrm{d}x^j} - \frac{\partial_c g^{m-k+a}_j}{\partial y^r} \right) \bar{\omega}^r \wedge \mathrm{d}x^j \\
&\quad + \left(G^a_s \frac{\partial g^s_j}{\partial z^J} - \frac{\partial g^{m-k+a}_j}{\partial z^J} \right) \mathrm{d}z^J \wedge \mathrm{d}x^j - \frac{\partial_c G^a_s}{\partial y^r} \bar{\omega}^r \wedge \bar{\omega}^s - \frac{\partial G^a_s}{\partial z^J}\, \mathrm{d}z^J \wedge \bar{\omega}^s \\
&\quad + \left(G^a_s \frac{\partial g^s_j}{\partial y^{m-k+b}} - \frac{\partial g^{m-k+a}_j}{\partial y^{m-k+b}} \right) \bar{\varphi}^b \wedge \mathrm{d}x^j - \frac{\partial G^a_s}{\partial y^{m-k+b}} \bar{\varphi}^b \wedge \bar{\omega}^s,
\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}
\bar{p}_1\, \mathrm{d}\varphi^a = &\left(G^a_s \frac{\partial_c g^s_j}{\partial y^r} + \frac{\mathrm{d}_c G^a_r}{\mathrm{d}x^j} - \frac{\partial_c g^{m-k+a}_j}{\partial y^r} \right) \bar{\omega}^r \wedge \mathrm{d}x^j \\
&+ \left(G^a_s \frac{\partial g^s_j}{\partial z^J} - \frac{\partial g^{m-k+a}_j}{\partial z^J} \right) \hat{\omega}^J \wedge \mathrm{d}x^j \\
&+ \left(G^a_s \frac{\partial g^s_j}{\partial y^{m-k+b}} - \frac{\partial g^{m-k+a}_j}{\partial y^{m-k+b}} \right) \bar{\varphi}^b \wedge \mathrm{d}x^j.
\end{aligned} \tag{3.50}$$

Since for each a the φ^a is contact, $\mathrm{d}\varphi^a$ is also contact. Hence

$$\begin{aligned}
\bar{h}\, \mathrm{d}\varphi^a &= \left(G^a_s \frac{\mathrm{d}'_c g^s_j}{\mathrm{d}x^i} - \frac{\mathrm{d}'_c g^{m-k+a}_j}{\mathrm{d}x^i} + \left(G^a_s \frac{\partial g^s_j}{\partial z^J} - \frac{\partial g^{m-k+a}_j}{\partial z^J} \right) z^J_i \right) \mathrm{d}x^i \wedge \mathrm{d}x^j \\
&= \left(G^a_s \frac{\mathrm{d}_c g^s_j}{\mathrm{d}x^i} - \frac{\mathrm{d}_c g^{m-k+a}_j}{\mathrm{d}x^i} \right) \mathrm{d}x^i \wedge \mathrm{d}x^j = 0,
\end{aligned} \tag{3.51}$$

and we can see that (3.46) holds. We also get (3.47) and

[2] With theorem 3.9, some results previously obtained in [16, 20] are simplified. In particular, formula (3.48) is significant, saying that all the functions $\mathcal{C}^a_{Jj}$ vanish. Due to this identity, some important formulae, e.g. constrained Poincaré–Cartan form or constrained equations of motion, take a more friendly coordinate form.

$$\left(G^a_s \frac{\partial g^s_j}{\partial z^J} - \frac{\partial g^{m-k+a}_j}{\partial z^J}\right) z^J_i - \left(G^a_s \frac{\partial g^s_i}{\partial z^J} - \frac{\partial g^{m-k+a}_i}{\partial z^J}\right) z^J_j = 0 \qquad \forall i, j, \tag{3.52}$$

meaning that (3.48) are true.

Substituting (3.48) into (3.50), we obtain the first assertion of theorem 3.9.

Finally, if $\varphi \in \mathcal{C}^0$, we have $\varphi = F_a \varphi^a$, hence $\mathrm{d}\varphi = F_a\, \mathrm{d}\varphi^a + \mathrm{d}F_a \wedge \varphi^a$, and (since φ^a are contact forms) $\bar{p}_1\, \mathrm{d}\varphi = F_a \bar{p}_1\, \mathrm{d}\varphi^a + \bar{h}\, \mathrm{d}F_a \wedge \varphi^a$. Assertion (2) now follows from (1). □

For convenience, let us write

$$\mathcal{E}'^{\mathrm{c}}_s\left(g^{m-k+a}_j\right) = \frac{\partial_{\mathrm{c}} g^{m-k+a}_j}{\partial y^s} - \frac{\mathrm{d}'_{\mathrm{c}} G^a_s}{\mathrm{d}x^j} - \frac{\partial_{\mathrm{c}} g^r_j}{\partial y^s} G^a_r, \tag{3.53}$$

$$\mathcal{E}^{\mathrm{c}}_s\left(g^{m-k+a}_j\right) = \frac{\partial_{\mathrm{c}} g^{m-k+a}_j}{\partial y^s} - \frac{\mathrm{d}_{\mathrm{c}} G^a_s}{\mathrm{d}x^j} - \frac{\partial_{\mathrm{c}} g^r_j}{\partial y^s} G^a_r. \tag{3.54}$$

With this notation,

$$\begin{aligned} \mathrm{d}\varphi^a &= -\mathcal{E}'^{\mathrm{c}}_r\left(g^{m-k+a}_j\right)\bar{\omega}^r \wedge \mathrm{d}x^j - \frac{\partial_c G^a_s}{\partial y^r}\bar{\omega}^r \wedge \bar{\omega}^s - \frac{\partial G^a_s}{\partial z^J}\,\mathrm{d}z^J \wedge \bar{\omega}^s \\ &\quad + \left(G^a_s \frac{\partial g^s_j}{\partial y^{m-k+b}} - \frac{\partial g^{m-k+a}_j}{\partial y^{m-k+b}}\right)\bar{\varphi}^b \wedge \mathrm{d}x^j - \frac{\partial G^a_s}{\partial y^{m-k+b}}\bar{\varphi}^b \wedge \bar{\omega}^s, \end{aligned} \tag{3.55}$$

$$\bar{p}_1\, \mathrm{d}\varphi^a = -\mathcal{E}^{\mathrm{c}}_r\left(g^{m-k+a}_j\right)\bar{\omega}^r \wedge \mathrm{d}x^j + \left(G^a_s \frac{\partial g^s_j}{\partial y^{m-k+b}} - \frac{\partial g^{m-k+a}_j}{\partial y^{m-k+b}}\right)\bar{\varphi}^b \wedge \mathrm{d}x^j, \tag{3.56}$$

and $\mathrm{d}\varphi^a \wedge \omega_j = \psi^a_j$ + 2-contact form + constraint form, where

$$\psi^a_j = -\mathcal{E}'^{\mathrm{c}}_s\left(g^{m-k+a}_j\right)\bar{\omega}^s \wedge \omega_0 + \frac{\partial G^a_s}{\partial z^J}\bar{\omega}^s \wedge \mathrm{d}z^J \wedge \omega_j, \tag{3.57}$$

i.e.,

$$\bar{p}_1 \psi^a_j = -\mathcal{E}^{\mathrm{c}}_s\left(g^{m-k+a}_j\right)\bar{\omega}^s \wedge \omega_0. \tag{3.58}$$

The above formulae will be useful later when we shall study constrained variations.

Remark 3.10. Theorem 3.9, item (1) or (2), is a rather surprising property of nonholonomic constraints: in fact it claims that the canonical distribution of a regular nonholonomic constraint is *optimal* in the sense that $\mathcal{C}^0$, hence *the constraint ideal, is generated by Lepage forms*. This property can be viewed as an important *intrinsic definition* of the canonical distribution and the constraint ideal. Although constraint 1-forms φ^a are of local nature (as local generators of the codistribution $\mathcal{C}^0$), their intrinsic characterization is that they are *Lepage forms*; the horizontal components of these forms are then *Lagrangians* for the constraint Q (for more details we refer to [16], the concept of the Lagrangian constraint).

4. The first variation formula on a nonholonomic constraint

Consider a nonholonomic constraint $Q \subset J^1Y$. Our aim is to propose a variational principle *on the constraint submanifold Q, providing Chetaev-reduced equations as equations for 'constrained extremals'*.

It is important to note that we have to distinguish *two distinct cases* as follows:

- A variational principle for a constrained to Q Lagrangian system that is *originally defined on* J^1Y;

- A 'true' variational principle on Q: the integrand of the action is a differential form *on* Q: it need not come from an 'unconstrained' Lagrangian system on the surrounding manifold J^1Y.

4.1. Admissible sections, nonholonomic virtual displacements

Admissible sections are solutions of the equations of the constraint. This means that they have to end in the constraint submanifold $Q \subset J^1Y$, otherwise speaking, they are *sections of the fibred manifold* $\bar{\pi}_1 : Q \to X$. Every admissible section δ has a counterpart in Y: it is a section γ of $\pi : Y \to X$, given by

$$\gamma = \bar{\pi}_{1,0} \circ \delta. \tag{4.1}$$

Often we are interested in *holonomic admissible sections* (i.e., $\delta = J^1\gamma$). In this context we also speak about *admissible sections of* $\pi : Y \to X$: a section γ of π is called admissible for Q if $J^1\gamma$ is admissible.

We can immediately see that the following proposition holds.

Proposition 4.1. *Given a regular nonholonomic constraint $Q \subset J^1Y$, holonomic admissible sections are integral sections of the canonical distribution $\mathcal{C}$ on Q. Consequently, for every admissible section γ of π, $J^1\gamma$ is an integral section of $\mathcal{C}$.*

Proof. If γ is a section of π such that $\mathrm{Im} J^1\gamma \subset Q$, then for all constraint 1-forms φ^a, $1 \leqslant a \leqslant k$, we obtain $J^1\gamma^*\varphi^a = 0$, since $J^1\gamma^*\bar{\omega}^\sigma = 0$ for all $1 \leqslant \sigma \leqslant m$. □

A correct concept of *admissible variations* is, contrary to the unconstrained case, difficult and not so straightforward: the key to this concept is the *canonical distribution*: as pointed out in [14], *nonholonomic virtual displacements*, or *admissible variations are realized by Chetaev vector fields.*[3]

It is important to note that to obtain variations (deformations) of admissible sections one has to consider Chetaev vector fields that are *projectable onto X*. Indeed, variations of a *section* of Q induced by a projectable vector field provide a 1-parametric family of maps that all are *sections* of the constraint manifold. Precisely, if Z is a $\bar{\pi}_1$-projectable vector field belonging to the canonical distribution, and ϕ_u, respectively ϕ_{0u} is the local 1-parameter group of Z, respectively of the $\bar{\pi}_1$-projection of Z, then for every parameter $u \in (-\epsilon, \epsilon)$ from an appropriate ϵ-neighbourhood of $0 \in \mathbb{R}$, the composed mapping

$$\delta_u = \phi_u \delta \phi_{0u}^{-1} \tag{4.2}$$

is a *section* of $\bar{\pi}_1$. In this way we get a 1*-parameter family of admissible sections* $\{\delta_u\}$, induced by Z.

Looking at formulae (3.27) and (3.42) defining Chetaev vector fields, and having in mind theorems 3.4 and 3.8, we immediately realize a rather surprising property of nonholonomic variations. Namely, there is *no direct concept of nonholonomic variations of an admissible section γ of π*, unless the canonical distribution is projectable onto Y. In the latter 'simple' situation, we have a proposition as follows:

Proposition 4.2. *Let $Q \subset J^1Y$ be a regular nonholonomic constraint. Assume that the canonical distribution $\mathcal{C}$ is $\bar{\pi}_{1,0}$-projectable, denote $\mathcal{D}$ its projection. Let γ be an admissible*

[3] Note that, nonholonomic 'virtual displacements' take place in the manifold Q, i.e. in the space of events and constrained velocities.

section of π. Given a π-projectable vector field $\xi \in \mathcal{D}$ with the local 1-parameter group ψ_u, $u \in (-\epsilon, \epsilon)$, then

$$\gamma_u = \psi_u \gamma \psi_{0u}^{-1}, \qquad u \in (-\epsilon, \epsilon) \tag{4.3}$$

is a 1-parameter family of sections of $\pi : Y \to X$ such that for every u, γ_u is the projection of an admissible section $X \to Q$.

If Q is semiholonomic then (4.3) is a 1-parameter family of admissible sections of $\pi : Y \to X$.

Proof. Let $\xi \in \mathcal{D}$, and choose $Z \in \mathcal{C}$ such that ξ is the $\bar{\pi}_{1,0}$-projection of Z. Denote ϕ_u, $u \in (-\epsilon, \epsilon)$ the 1-parameter group of Z. Then

$$\psi_u \circ \bar{\pi}_{1,0} = \bar{\pi}_{1,0} \circ \phi_u, \qquad \text{and} \qquad \phi_{0u} = \psi_{0u}. \tag{4.4}$$

Consider the deformation of the section $J^1\gamma$ by Z, i.e. the family of sections

$$\delta_u = \phi_u \circ J^1\gamma \circ \phi_{0u}^{-1}, \qquad u \in (-\epsilon, \epsilon). \tag{4.5}$$

Then for every u, the projection of δ_u takes the form

$$\gamma_u = \bar{\pi}_{1,0} \circ \delta_u = \bar{\pi}_{1,0} \circ \phi_u \circ J^1\gamma \circ \phi_{0u}^{-1} = \psi_u \circ \bar{\pi}_{1,0} \circ J^1\gamma \circ \psi_{0u}^{-1} = \psi_u \circ \gamma \circ \psi_{0u}^{-1}, \tag{4.6}$$

as desired.

If Q is semiholonomic, we have $J^1\gamma_u = J^1\big(\psi_u \gamma \psi_{0u}^{-1}\big) = J^1\psi_u \circ J^1\gamma \circ \psi_{0u}^{-1}$. Since $J^1\gamma$ is a section of $\bar{\pi}_1$ and the vector field $J^1_c\xi$ belongs to the canonical distribution $\mathcal{C}$, so that it is tangent to the manifold Q, we get for all points x from the domain of γ and for all $u \in (-\epsilon, \epsilon)$ that $J^1\gamma_u(x) = J^1\psi_u\big(J^1\gamma\big(\psi_{0u}^{-1}(x)\big)\big) \in Q$. This means that all γ_u are admissible sections. □

If the canonical distribution is *not projectable* onto Y, taking an admissible section γ of π and a 'variation vector field' $Z \in \mathcal{C}$, we get a family of admissible sections of the constraint Q

$$\delta_u = \phi_u J^1\gamma \phi_{0u}^{-1}. \tag{4.7}$$

First of all, sections δ_u of this family *need not be holonomic* (i.e., a deformation (variation) of prolongation of an admissible section of π need not correspond to a prolongation of a section of π) which is a violation of the 'classical' principle of virtual displacements. Moreover, the projection of $\{\delta_u\}$, i.e. the family of sections of π of the form

$$\gamma_u = \bar{\pi}_{1,0}\phi_u J^1\gamma \phi_{0u}^{-1} \tag{4.8}$$

need not be induced by a vector field on Y.

4.2. The nonholonomic first variation formula for ambient Lagrangian systems

Given a nonholonomic constraint $Q \subset J^1Y$ and a Lagrangian λ on J^1Y we shall introduce a variational principle for sections of the constraint Q such that the extremals of the constraint action are solutions of the reduced nonholonomic equations.

First, let us summarize main points to be considered:

- The variational principle is formulated for the fibred manifold $\bar{\pi}_1 : Q \to X$, $\dim X = n$, endowed with the canonical distribution $\mathcal{C}$.
- Admissible paths are sections of the fibred manifold $\bar{\pi}_1 : Q \to X$.
- Admissible variations are $\bar{\pi}_1$-projectable vector fields belonging to the canonical distribution (Chetaev vector fields).

It remains to specify *the integrand of the action function*. In this section we shall assume that the constrained system arises from an unconstrained Lagrangian system on J^1Y.

As above, $\iota : Q \to J^1Y$ is the canonical embedding of a nonholonomic constraint in J^1Y, $\mathcal{C}$ is the canonical distribution on Q. Given a Lagrangian λ on J^1Y, it is known that the corresponding constrained Lagrangian system *is not* simply the horizontal n-form $\iota^*\lambda$ (see e.g. [14, 15]).

Definition 4.3. *By a constrained (to Q) system defined by a Lagrangian λ on J^1Y we shall mean the differential n-form $\iota^*\rho$, defined on Q, where ρ is a Lepage equivalent of λ.*

As we know, for $\dim X = 1$ the only choice is $\rho = \Theta_\lambda$ (the Cartan form), while for $\dim X > 1$ we have $\rho = \Theta_\lambda + \mathrm{d}\nu + \mu$, where Θ_λ is the Poincaré–Cartan form, ν is a contact form and μ is a 2-contact form; the latter forms need not be determined by the Lagrangian.

For the sake of clarity it is worth considering the case $\dim X = 1$ (constrained curves) and $\dim X > 1$ (constrained fields) separately.

Mechanics. Let $\dim X = 1$. Given a Lagrangian λ on J^1Y and a nonholonomic constraint Q in J^1Y, there arises *a unique constrained system $\iota^*\Theta_\lambda$ defined on Q*.

Let us recall a useful relation between $\iota^*\Theta_\lambda$, the constrained to Q Cartan form of λ, and $\Theta_{\iota^*\lambda}$, the (local)[4] Cartan form of the constrained Lagrangian [15]:

Proposition 4.4. *Let us write*

$$\bar{\lambda} = \iota^*\lambda = (L \circ \iota)\,\mathrm{d}t = \bar{L}\,\mathrm{d}t, \tag{4.9}$$

$$\Theta_{\bar{\lambda}} = \Theta_{\iota^*\lambda} = \bar{L}\,\mathrm{d}t + \sum_{l=1}^{m-k} \frac{\partial \bar{L}}{\partial \dot{q}^l}\bar{\omega}^l, \tag{4.10}$$

and

$$L_a = \frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota, \qquad 1 \leqslant a \leqslant k. \tag{4.11}$$

The form $\iota^*\Theta_\lambda$ locally splits into two terms as follows:

$$\iota^*\Theta_\lambda = \Theta_{\bar{\lambda}} + L_a\varphi^a, \tag{4.12}$$

i.e. the difference $\iota^*\Theta_\lambda - \Theta_{\bar{\lambda}}$ is a constraint form.

Definition 4.5. *Let $\Omega \subset X$ be a piece of X (for simplicity we can take $\Omega = [a, b]$, $a < b$). Denote by $\mathcal{S}_\Omega(\bar{\pi}_1)$ the set of sections of the projection $\bar{\pi}_1 : Q \to X$, whose domains are neighbourhoods of Ω. The function*

$$\mathcal{S}_\Omega(\bar{\pi}_1) \ni \delta \to \int_\Omega \delta^*\iota^*\Theta_\lambda \in \mathbb{R}, \tag{4.13}$$

will be called the constrained (to Q) action function of the Lagrangian λ over Ω.

Let Z be a $\bar{\pi}_1$-projectable vector field belonging to the canonical distribution, and $\{\phi_u\}$, respectively $\{\phi_{0u}\}$ the local 1-parameter group of Z, respectively of the $\bar{\pi}_1$-projection of Z. Given a section $\delta \in \mathcal{S}_\Omega(\bar{\pi}_1)$, we get for every u (from an appropriate ϵ-neighbourhood of $0 \in \mathbb{R}$) a deformed section $\delta_u = \phi_u\delta\phi_{0u}^{-1}$ of $\bar{\pi}_1$ defined in a neighbourhood of $\phi_{0u}(\Omega)$; the 1-parameter family $\{\delta_u\}$ is called *constrained variation of* δ induced by Z.

In this way, every $\bar{\pi}_1$-projectable Chetaev vector field Z induces a real-valued function

$$u \to \int_{\phi_{0u}(\Omega)} \delta_u^*\iota^*\Theta_\lambda \in \mathbb{R}. \tag{4.14}$$

[4] Note that the 'Cartan form of the constrained Lagrangian', $\Theta_{\iota^*\lambda}$, need not be globally defined on Q.

Differentiating at $u = 0$ we get the following function on the set $\mathcal{S}_\Omega(\bar{\pi}_1)$:

$$\mathcal{S}_\Omega(\bar{\pi}_1) \ni \delta \to \left(\frac{\mathrm{d}}{\mathrm{d}u}\int_{\phi_{0u}(\Omega)} \delta_u^* \iota^* \Theta_\lambda\right)_{u=0} = \int_\Omega \delta^* \mathcal{L}_Z \iota^* \Theta_\lambda \in \mathbb{R}; \tag{4.15}$$

we shall call it the *first constrained variation of the action function* of λ over Ω, *induced by* Z.

Remark 4.6. Applying the same procedure to the 1-form $\mathcal{L}_Z \iota^* \Theta_\lambda$ in place of $\iota^*\Theta_\lambda$ we obtain the *second constrained variation of the action function* of λ over Ω as the second Lie derivative of $\iota^*\Theta_\lambda$, and in the same way we easily get *higher constrained variations* (induced, in general, by different Chetaev vector fields).

Let us turn back to the first constrained variation. To study *constrained sections of the fibred manifold* $\pi : Y \to X$, we have to restrict the domain of definition $\mathcal{S}_\Omega(\bar{\pi}_1)$ of the function (4.15) to the subset $\mathcal{S}_\Omega^h(\bar{\pi}_1)$ of *holonomic* sections of the projection $\bar{\pi}_1$, i.e. to sections of $\bar{\pi}_1 : Q \to X$ of the form $\delta = J^1\gamma$ where $\gamma \in \mathcal{S}_\Omega(\pi)$. Then the first constrained variation (4.15) can be regarded as a function

$$\mathcal{S}_{\Omega,Q}(\pi) \ni \gamma \to \int_\Omega J^1\gamma^* \mathcal{L}_Z \iota^* \Theta_\lambda \in \mathbb{R} \tag{4.16}$$

defined on a *subset of sections of the projection* $\pi : Y \to X$.

Remark 4.7. It should be stressed that the restricted first constrained variation cannot be obtained via a 'variation procedure' from a certain 'action' defined on the set $\mathcal{S}_{\Omega,Q}(\pi)$ (see proposition 4.2 and the discussion around).

Applying to (4.16) Cartan's formula for the decomposition of Lie derivative, and keeping notations introduced so far, we obtain the following theorem:

Theorem 4.8. *Let λ be a Lagrangian on J^1Y. Given a nonholonomic constraint $\iota : Q \to J^1Y$, the constrained first variation formula takes the form*

$$\int_\Omega J^1\gamma^* \mathcal{L}_Z \iota^* \Theta_\lambda = \int_\Omega J^1\gamma^* i_Z \iota^* \,\mathrm{d}\Theta_\lambda + \int_\Omega J^1\gamma^* \,\mathrm{d}i_Z \iota^* \Theta_\lambda, \tag{4.17}$$

or, has an equivalent expression

$$\int_\Omega J^1\gamma^* \mathcal{L}_Z \iota^* \Theta_\lambda = \int_\Omega J^1\gamma^* i_Z (\mathrm{d}\Theta_{\bar{\lambda}} + L_a \psi^a) + \int_\Omega J^1\gamma^* \,\mathrm{d}i_Z \Theta_{\bar{\lambda}}, \tag{4.18}$$

where Z is any $\bar{\pi}_1$-projectable Chetaev vector field.

Proof. First we show that (4.17) and (4.18) are equivalent. From (4.12) we obtain

$$\iota^*\mathrm{d}\Theta_\lambda = \mathrm{d}\Theta_{\bar{\lambda}} + L_a \psi^a + \text{ 2-contact form} + \text{ constraint form}, \tag{4.19}$$

where the 2-forms ψ^a were defined in (3.36). Applying to this formula contraction by Z and the pullback by $J^1\gamma$, we can see that the last two terms vanish. Indeed, the contraction of a constraint form by a Chetaev vector field Z is a constraint form, hence contact, and similarly a contraction of a 2-contact form (by any vector field) is a contact form, vanishing along prolongations of γ. Finally, (4.12) and $i_Z \varphi^a = 0 \,\forall\, a$, gives us $i_Z \iota^* \Theta_\lambda = i_Z \Theta_{\bar{\lambda}}$.

It remains to prove that (4.17), respectively (4.18), represents a decomposition into a 'constrained Euler–Lagrange term' and a boundary term. The latter is obvious, since

$$\int_\Omega J^1\gamma^* \mathrm{d}i_Z \iota^* \Theta_\lambda = \int_{\partial\Omega} J^1\gamma^* i_Z \iota^* \Theta_\lambda. \tag{4.20}$$

Hence, we have only to show that the term $J^1\gamma^* i_Z \iota^* \mathrm{d}\Theta_\lambda$, or, equivalently, the horizontal part of $i_Z \iota^* \mathrm{d}\Theta_\lambda$, does not depend upon the 'first-order components' of the vector field Z, i.e. the components $\tilde{Z}^s$ at $\partial/\partial \dot{q}^s$, $1 \leqslant s \leqslant m-k$. This is easily seen from (4.18). Indeed,

$$\bar{h} i_Z(\mathrm{d}\Theta_{\bar{\lambda}} + L_a \psi^a) = \bar{h} i_Z \bar{p}_1(\mathrm{d}\Theta_{\bar{\lambda}} + L_a \psi^a) = \left(\mathcal{E}_s^{\mathrm{c}}(\bar{L}) - L_a \mathcal{E}_s^{\mathrm{c}}(g^{m-k+a})\right) Z^s \,\mathrm{d}t, \tag{4.21}$$

proving that the term

$$\int_\Omega J^1\gamma^* i_Z \iota^* \mathrm{d}\Theta_\lambda = \int_\Omega J^1\gamma^* i_Z(\mathrm{d}\Theta_{\bar{\lambda}} + L_a \psi^a) \tag{4.22}$$

really has the meaning of a 'constrained Euler–Lagrange term'. □

Remark 4.9. The vector field Z need not be a symmetry of the (induced) contact ideal on Q; hence $\mathcal{L}_Z$ need not be compatible with the decomposition of the 1-form $\iota^*\Theta_\lambda$ into the horizontal and contact component. Indeed, in general,

$$\mathcal{L}_Z \bar{h} \iota^* \Theta_\lambda = \mathcal{L}_Z \iota^* \lambda \neq \bar{h} \mathcal{L}_Z \iota^* \Theta_\lambda \tag{4.23}$$

so that on the left-hand side of the constrained first variation formula one cannot put the Lie derivative of the 'constrained Lagrangian', $\mathcal{L}_Z \bar{\lambda}$, instead of $\mathcal{L}_Z \iota^* \Theta_\lambda$.

Let us compute the term that appears as the horizontal part of $\mathcal{L}_Z \iota^* \Theta_\lambda$; hence substitutes the role of a 'transformed constrained Lagrangian': taking into account that Z is a Chetaev vector field ($i_Z \varphi^a = 0 \,\forall\, a$) and the constraint forms φ^a are contact, we obtain up to a contact form (indicated by dots)

$$\begin{aligned}
\mathcal{L}_Z \iota^* \Theta_\lambda &= \mathcal{L}_Z \Theta_{\iota^*\lambda} + \mathcal{L}_Z (L_a \varphi^a) \\
&= \mathcal{L}_Z \iota^* \lambda + \frac{\partial (L\circ\iota)}{\partial \dot{q}^l} \mathcal{L}_Z \bar{\omega}^l + L_a \mathcal{L}_Z \varphi^a + \cdots \\
&= \mathcal{L}_Z \iota^* \lambda + \frac{\partial (L\circ\iota)}{\partial \dot{q}^l} (i_Z \,\mathrm{d}\bar{\omega}^l + \mathrm{d} i_Z \bar{\omega}^l) + L_a i_Z \,\mathrm{d}\varphi^a + \cdots \\
&= \mathcal{L}_Z \iota^* \lambda + \frac{\partial (L\circ\iota)}{\partial \dot{q}^l} (Z^0 \,\mathrm{d}\dot{q}^l - \tilde{Z}^l \,\mathrm{d}t + d(Z^l - \dot{q}^l Z^0)) + L_a i_Z \psi^a + \cdots.
\end{aligned} \tag{4.24}$$

After splitting to the horizontal and contact component we finally obtain

$$\bar{h} \mathcal{L}_Z \iota^* \Theta_\lambda = \mathcal{L}_Z \bar{\lambda} - \left(L_a \mathcal{E}_l^{\mathrm{c}}(g^{m-k+a})(Z^l - \dot{q}^l Z^0) - \frac{\partial (L\circ\iota)}{\partial \dot{q}^l} \left(\frac{\mathrm{d_c} Z^l}{\mathrm{d}t} - \dot{q}^l \frac{\mathrm{d_c} Z^0}{\mathrm{d}t} - \tilde{Z}^l \right)\right) \mathrm{d}t \tag{4.25}$$

(note that the form above is defined on $\hat{Q} \subset J^2 Y$).

Hence, we can conclude:

Proposition 4.10. *With the notation $\iota^*\lambda = \bar{\lambda} = \bar{L}\,\mathrm{d}t$ the integrand on the left-hand side of the constrained first variation formula (4.17), resp. (4.18) reads*

$$\begin{aligned}
J^1\gamma^* \mathcal{L}_Z \iota^* \Theta_\lambda = J^2\gamma^* \Bigg[\mathcal{L}_Z \bar{\lambda} - \Bigg(& L_a \mathcal{E}_l^{\mathrm{c}}(g^{m-k+a})(Z^l - \dot{q}^l Z^0) \\
& - \frac{\partial \bar{L}}{\partial \dot{q}^l} \left(\frac{\mathrm{d_c} Z^l}{\mathrm{d}t} - \dot{q}^l \frac{\mathrm{d_c} Z^0}{\mathrm{d}t} - \tilde{Z}^l \right) \Bigg) \Bigg] \mathrm{d}t.
\end{aligned} \tag{4.26}$$

If, in particular, the constrained variation Z is a contact symmetry then

$$\int_\Omega J^1\gamma^* \mathcal{L}_Z \iota^* \Theta_\lambda = \int_\Omega J^1\gamma^* \mathcal{L}_Z \bar{\lambda}. \tag{4.27}$$

Remark 4.11. If Q is semiholonomic then, as we have seen, every constrained variation $Z \in \mathcal{C}$ such that $\xi = T\bar{\pi}_{1,0} \cdot Z \neq 0$, is a contact symmetry, moreover, $Z = J^1_c\xi$. Hence,

$$\int_\Omega J^1\gamma^* \mathcal{L}_{J^1_c\xi}\iota^*\Theta_\lambda = \int_\Omega J^1\gamma^*\mathcal{L}_{J^1_c\xi}\bar{\lambda} \tag{4.28}$$

for every π-projectable vector field $\xi \in \mathcal{D}$, where $\mathcal{D}$ is the projection of the canonical distribution.

Field theory. Now, let $\dim X = n > 1$. Consider a nonholonomic constraint $\iota : Q \to J^1Y$, $\operatorname{codim} Q = \kappa$, and its canonical distribution $\mathcal{C}$. To avoid technical problems with points of discontinuity of generators of $\mathcal{C}$, in what follows, we assume the constraint Q be *regular*, and put $\operatorname{rank}\mathcal{C} = k$, where k is a constant, $1 \leqslant k < m$.

Similarly as in mechanics, we have a local splitting of the constrained Cartan form $\iota^*\Theta_\lambda$ of a Lagrangian λ on J^1Y, as follows (cf [16] and (3.48)):

$$\iota^*\Theta_\lambda = \Theta_{\bar{\lambda}} + L^j_a\varphi^a \wedge \omega_j, \tag{4.29}$$

where

$$\Theta_{\bar{\lambda}} = \bar{L}\omega_0 + \frac{\partial \bar{L}}{\partial z^J}\frac{\partial z^J}{\partial y^s_j}\bar{\omega}^s \wedge \omega_j \tag{4.30}$$

is the local 'Cartan form' of the constrained Lagrangian $\iota^*\lambda = \bar{\lambda}$, and

$$L^j_a = \frac{\partial L}{\partial y^{m-k+a}_j} \circ \iota. \tag{4.31}$$

Definition 4.12. *Let λ be a Lagrangian on J^1Y, ρ its Lepage equivalent. The constrained action function of ρ over a piece Ω of X is a real function*

$$\mathcal{S}_\Omega(\bar{\pi}_1) \ni \delta \to \int_\Omega \delta^*\iota^*\rho \in \mathbb{R}. \tag{4.32}$$

Consider a $\bar{\pi}_1$-projectable Chetaev vector field Z on Q. Given a section $\delta \in \mathcal{S}_\Omega(\bar{\pi}_1)$, consider its *constrained variation induced by Z*,

$$\delta_u = \phi_u\delta\phi_{0u}, \qquad u \in (-\epsilon, \epsilon) \tag{4.33}$$

and the induced function

$$u \to \int_{\phi_{0u}(\Omega)} \delta^*_u\iota^*\rho \in \mathbb{R}. \tag{4.34}$$

Differentiating at $u = 0$ as usual, we get the *first constrained variation*[5] *of the action function* of ρ over Ω, *induced by Z*; it reads

$$\mathcal{S}_\Omega(\bar{\pi}_1) \ni \delta \to \left(\frac{\mathrm{d}}{\mathrm{d}u}\int_{\phi_{0u}(\Omega)} \delta^*_u\iota^*\rho\right)_{u=0} = \int_\Omega \delta^*\mathcal{L}_Z\iota^*\rho \in \mathbb{R}. \tag{4.35}$$

Restricting to *constrained sections of the fibred manifold* $\pi : Y \to X$, i.e., restricting the domain of definition $\mathcal{S}_\Omega(\bar{\pi}_1)$ of (4.35) to the subset of *holonomic* sections of the projection $\bar{\pi}_1$, we get the function

$$\mathcal{S}_{\Omega,Q}(\pi) \ni \gamma \to \int_\Omega J^1\gamma^*\mathcal{L}_Z\iota^*\rho \in \mathbb{R}, \tag{4.36}$$

[5] Again, higher constrained variations are easily obtained: applying the same procedure (with eventually another Chetaev vector field) to the n-form $\mathcal{L}_Z\iota^*\rho$ in place of $\iota^*\rho$ we obtain the *second constrained variation of the action function* as the second Lie derivative of $\iota^*\rho$, and so on for higher constrained variations.

defined on a *subset of sections of the projection* $\pi : Y \to X$. Keep in mind that the restricted first variation of the action usually(!) does not come from a variation of action defined on sections of π.

The decomposition of the Lie derivative gives us the constrained first variation formula for a Lepage form ρ. However, to a Lagrangian λ, we have a family of Lepage equivalents. We know that for an unconstrained first-order Lagrangian, the first variation formula does not depend upon a choice of its Lepage equivalent. Hence, we also wish to clarify the impact of nonuniqueness of ρ in the presence of constraints. The results are summarized as follows:

Theorem 4.13. *Let λ be a Lagrangian on J^1Y. Given a nonholonomic constraint $\iota : Q \to J^1Y$, for every $\bar{\pi}_1$-projectable Chetaev vector field Z on Q the constrained first variation formula reads*

$$\int_\Omega J^1\gamma^*\mathcal{L}_Z\iota^*\rho = \int_\Omega J^1\gamma^* i_Z\iota^*\mathrm{d}\rho + \int_{\partial\Omega} J^1\gamma^* i_Z\iota^*\rho, \tag{4.37}$$

where ρ is a Lepage equivalent of λ.

Moreover, the first term on the right-hand side of (4.37) does not depend upon a choice of a Lepage equivalent of λ, and the other terms depend only upon the at most 1-contact part $\theta = \Theta_\lambda + p_1\,\mathrm{d}\nu$ of ρ, i.e., (4.37) is the same as

$$\int_\Omega J^1\gamma^*\mathcal{L}_Z\iota^*\theta = \int_\Omega J^1\gamma^* i_Z\iota^*\mathrm{d}\Theta_\lambda + \int_{\partial\Omega} J^1\gamma^* i_Z\iota^*\theta, \tag{4.38}$$

where Θ_λ is the Poincaré–Cartan form of λ.

The constrained first variation formula has also the following equivalent form

$$\int_\Omega J^1\gamma^*\mathcal{L}_Z\iota^*\theta = \int_\Omega J^1\gamma^* i_Z\big(\mathrm{d}\Theta_{\bar{\lambda}} + L_a^j\psi_j^a\big) + \int_{\partial\Omega} J^1\gamma^* i_Z(\Theta_{\bar{\lambda}} + \bar{p}_1\,\mathrm{d}\iota^*\nu). \tag{4.39}$$

Proof. First, we show the independence of (4.37) upon a choice of the at least 2-contact part of ρ. Every Lepage equivalent of a first-order Lagrangian λ reads $\rho = \Theta_\lambda + \mathrm{d}\nu + \mu$, where ν is a contact and μ is at least a 2-contact form on J^1Y. Then $\mathcal{L}_Z\iota^*\rho = \mathcal{L}_Z\iota^*\Theta_\lambda + \mathcal{L}_Z\iota^*\mathrm{d}\nu + \mathcal{L}_Z\iota^*\mu$. However, $\iota^*\mu$ is a 2-contact form on Q; hence the Lie derivative of $\iota^*\mu$ by any vector field yields a form that is at least 1-contact. This means that $\mathcal{L}_Z\iota^*\mu$ vanishes along $J^1\gamma$, so that

$$\int_\Omega J^1\gamma^*\mathcal{L}_Z\iota^*\rho = \int_\Omega J^1\gamma^*\mathcal{L}_Z\iota^*\theta = \int_\Omega J^1\gamma^* i_Z\iota^*\mathrm{d}\theta + \int_{\partial\Omega} J^1\gamma^* i_Z\iota^*\theta. \tag{4.40}$$

Let us show that also the right-hand sides of formulae (4.37) and (4.38) are the same. For the first term we have

$$J^1\gamma^* i_Z\iota^*\mathrm{d}\rho = J^1\gamma^* i_Z\iota^*(\mathrm{d}\Theta_\lambda + \mathrm{d}\mu) = J^1\gamma^* i_Z\iota^*\mathrm{d}\Theta_\lambda, \tag{4.41}$$

since $i_Z(\iota^*\mathrm{d}\mu)$ is contact. The second term becomes $J^1\gamma^* i_Z\iota^*\rho = J^1\gamma^* i_Z\iota^*(\theta + \tilde{\mu})$, where $\tilde{\mu}$ is at least 2-contact. Contraction of $\tilde{\mu}$ is contact, and vanishes along $J^1\gamma$, hence $J^1\gamma^* i_Z\iota^*\rho = J^1\gamma^* i_Z\iota^*\theta$.

Now, let us prove equivalence of (4.38) and (4.39). By (4.29) and (3.57) we obtain $\iota^*\mathrm{d}\Theta_\lambda = \mathrm{d}\Theta_{\bar{\lambda}} + L_a^j\psi_j^a +$ constraint form +2-contact form. However, the contraction with a Chetaev vector field of the last two terms is a contact form, vanishing along $J^1\gamma$. Computing the second integrand we get $J^1\gamma^* i_Z\iota^*\theta = J^1\gamma^* i_Z(\Theta_{\bar{\lambda}} + L_a^j\varphi^a \wedge \omega_j + \iota^* p_1\,\mathrm{d}\nu)$, however, the form $L_a^j i_Z(\varphi^a \wedge \omega_j)$ is contact, and $\iota^* p_1\,\mathrm{d}\nu = \bar{p}_1 d\iota^*\nu$.

Finally, we have to show that in the constrained first variation formula the term $J^1\gamma^* i_Z\iota^*\mathrm{d}\Theta_\lambda$ does not depend upon the 'first-order components' of the vector field Z, i.e. the components at $\partial/\partial z^J$. This is, however, immediately seen from (4.39), if we look at the definition of the forms $\Theta_{\bar{\lambda}}$ and ψ_j^a. $\square$

We can conclude that *for different Lepage equivalents ρ_1, ρ_2 of a Lagrangian, constrained equations of motion are the same*, however, *conservation laws may be different*, and the difference depends upon the contact form $\theta_1 - \theta_2$.

Remark 4.14. Note that the above theorem indeed presents a correct constrained first variation formula, since the 'Euler–Lagrange term' does not depend upon the first-order components of the vector field Z, i.e. components at $\partial/\partial z^J$.

Besides the integral constraint first variation formula we have also its *differential version*, that we shall call *infinitesimal constraint first variation formula*. It obviously takes one of the following equivalent forms:

$$\bar{h}\mathcal{L}_z\iota^*\rho = \bar{h}i_Z\iota^*\,\mathrm{d}\rho + \bar{h}\,\mathrm{d}i_Z\iota^*\rho, \tag{4.42}$$

$$\bar{h}\mathcal{L}_z\iota^*\theta = \bar{h}i_Z\iota^*\,\mathrm{d}\Theta_\lambda + \bar{h}\,\mathrm{d}i_Z\iota^*\theta, \tag{4.43}$$

$$\bar{h}\mathcal{L}_z\iota^*\theta = \bar{h}i_Z\big(\mathrm{d}\Theta_{\bar{\lambda}}^- + L_a^j\psi_j^a\big) + \bar{h}\,\mathrm{d}i_Z(\Theta_{\bar{\lambda}}^- + \bar{p}_1\,\mathrm{d}\iota^*\nu). \tag{4.44}$$

Definition 4.15. *We define the constrained Euler–Lagrange form of λ as follows:*

$$\bar{E}_\lambda = \bar{p}_1\iota^*\,\mathrm{d}\Theta_\lambda. \tag{4.45}$$

It is immediately seen that it holds.

Proposition 4.16.

$$\bar{E}_\lambda = \iota^* p_1\,\mathrm{d}\Theta_\lambda = \iota^* E_\lambda. \tag{4.46}$$

Writing the constrained Euler–Lagrange form in fibred coordinates adapted to the constraint, we obtain the formulae

$$\bar{E}_\lambda = \bar{p}_1(\mathrm{d}\Theta_{\bar{\lambda}}^- + L_a^j\psi_j^a) = \sum_{s=1}^{m-k}\mathcal{E}_s^{\mathrm{c}}(\bar{L}, L_a)\omega^s \wedge \omega_0 + \text{ constraint form}, \tag{4.47}$$

where for $\dim X = 1$

$$\begin{aligned}\mathcal{E}_s^{\mathrm{c}}(\bar{L}, L_a) &= \mathcal{E}_s^{\mathrm{c}}(\bar{L}) - L_a\mathcal{E}_s^{\mathrm{c}}(g^{m-k+a})\\ &= \frac{\partial_{\mathrm{c}}\bar{L}}{\partial q^s} - \frac{\mathrm{d}_{\mathrm{c}}}{\mathrm{d}t}\frac{\partial\bar{L}}{\partial\dot{q}^s} - L_a\left(\frac{\partial_{\mathrm{c}}g^{m-k+a}}{\partial q^s} - \frac{\mathrm{d}_{\mathrm{c}}}{\mathrm{d}t}\frac{\partial g^{m-k+a}}{\partial\dot{q}^s}\right),\end{aligned} \tag{4.48}$$

and for $\dim X = n > 1$

$$\begin{aligned}\mathcal{E}_s^{\mathrm{c}}(\bar{L}, L_a^j) &= \mathcal{E}_s^{\mathrm{c}}(\bar{L}) - L_a^j\mathcal{E}_s^{\mathrm{c}}\big(g_j^{m-k+a}\big)\\ &= \frac{\partial_{\mathrm{c}}\bar{L}}{\partial y^s} - \frac{\mathrm{d}_{\mathrm{c}}}{\mathrm{d}x^j}\frac{\partial\bar{L}}{\partial y_j^s} - \frac{\partial_{\mathrm{c}}g_j^r}{\partial y^s}\frac{\partial\bar{L}}{\partial y_j^r} - L_a^j\left(\frac{\partial_{\mathrm{c}}g_j^{m-k+a}}{\partial y^s} - \frac{\mathrm{d}_{\mathrm{c}}G_s^a}{\mathrm{d}x^j} - \frac{\partial_{\mathrm{c}}g_j^r}{\partial y^s}G_r^a\right).\end{aligned} \tag{4.49}$$

Components of $\bar{E}_\lambda$ are called *constrained Euler–Lagrange expressions* of λ.

Finally, the constrained first variation formula can also be expressed with the help of the constrained Euler–Lagrange form, to read e.g. as follows:

$$\int_\Omega J^1\gamma^*\mathcal{L}_z\iota^*\theta = \int_\Omega J^2\gamma^* i_Z\bar{E}_\lambda + \int_{\partial\Omega} J^1\gamma^* i_Z\theta. \tag{4.50}$$

4.3. Nonholonomic Euler–Lagrange equations

With the constrained first variation formula it is easy to obtain equations for constrained extremals—the nonholonomic Euler–Lagrange equations.

First note that the concept of a *'fixed endpoints' variation* over a piece Ω of X is defined in complete analogy with the unconstrained case: it is a $\bar{\pi}_1$-vertical Chetaev vector field on Q with the support in $\bar{\pi}_1^{-1}(\Omega)$.

Given a constrained Lagrangian system $\iota^*\rho$ on Q, a section γ of $\pi : Y \to X$ is called its *extremal over* Ω, if $\mathrm{Im} J^1\gamma \subset Q$, and

$$\int_\Omega J^1\gamma^* \mathcal{L}_z \iota^*\rho = 0, \tag{4.51}$$

for every 'fixed endpoints' variation Z over Ω. γ is called an *extremal* of $\iota^*\rho$, if it is an extremal of $\iota^*\rho$ over every piece Ω of X such that $\Omega \subset \mathrm{Dom}\,\gamma$.

The following theorem gives both *intrinsic* and *coordinate* versions of equations for extremals of nonholonomic systems, i.e. of 'constrained Euler–Lagrange equations'.

Theorem 4.17. *Let λ be a Lagrangian on J^1Y, ρ its Lepage equivalent, $\iota : Q \to J^1Y$ a nonholonomic constraint. Let γ be a section of π such that $\mathrm{Im} J^1\gamma \subset Q$. The following conditions are equivalent:*

(1) γ is an extremal of the constrained system $\iota^*\rho$.

(2) For every $\bar{\pi}_1$-vertical Chetaev vector field Z on Q and any constraint $(n+1)$-form φ

$$J^1\gamma^* i_Z(\mathrm{d}\iota^*\rho + \varphi) = 0. \tag{4.52}$$

(3) For every $\bar{\pi}_1$-projectable Chetaev vector field Z on Q and any constraint $(n+1)$-form φ

$$J^1\gamma^* i_Z(\mathrm{d}\iota^*\rho + \varphi) = 0. \tag{4.53}$$

(4) For every Chetaev vector field Z on Q and any constraint $(n+1)$-form φ

$$J^1\gamma^* i_Z(\mathrm{d}\iota^*\rho + \varphi) = 0. \tag{4.54}$$

(5) The constrained Euler-Lagrange form $\bar{E}_\lambda$ vanishes along $J^2\gamma$, i.e.,

$$\bar{E}_\lambda \circ J^2\gamma = 0. \tag{4.55}$$

(6) γ satisfies the following system of differential equations

$$\left(\mathcal{E}^{\mathrm{c}}_s(\bar{L}) - L_a \mathcal{E}^{\mathrm{c}}_s(g^{m-k+a})\right) \circ J^2\gamma = 0, \tag{4.56}$$

i.e.,

$$\frac{\partial_{\mathrm{c}}\bar{L}}{\partial q^s} - \frac{\mathrm{d_c}}{\mathrm{d}t}\frac{\partial \bar{L}}{\partial \dot{q}^s} - L_a\left(\frac{\partial_{\mathrm{c}} g^{m-k+a}}{\partial q^s} - \frac{d_{\mathrm{c}}}{\mathrm{d}t}\frac{\partial g^{m-k+a}}{\partial \dot{q}^s}\right) = 0 \tag{4.57}$$

if $\dim X = 1$ *and*

$$\left(\mathcal{E}^{\mathrm{c}}_s(\bar{L}) - L^j_a \mathcal{E}^{\mathrm{c}}_s\left(g^{m-k+a}_j\right)\right) \circ J^2\gamma = 0, \tag{4.58}$$

i.e.,

$$\frac{\partial_{\mathrm{c}}\bar{L}}{\partial y^s} - \frac{\mathrm{d_c}}{\mathrm{d}x^j}\frac{\partial \bar{L}}{\partial y^s_j} - \frac{\partial_{\mathrm{c}} g^r_j}{\partial y^s}\frac{\partial \bar{L}}{\partial y^r_j} - L^j_a\left(\frac{\partial_{\mathrm{c}} g^{m-k+a}_j}{\partial y^s} - \frac{\mathrm{d_c} G^a_s}{\mathrm{d}x^j} - \frac{\partial_{\mathrm{c}} g^r_j}{\partial y^s} G^a_r\right) = 0 \tag{4.59}$$

if $\dim X = n > 1$*, respectively, together with the equations of the constraint.*

Equations (4.52)–(4.54) do not depend upon a choice of a Lepage equivalent ρ of λ, so that they may be equivalently written with Θ_λ in place of ρ.

Proof. The constrained first variation formula (4.37) gives us that γ is an extremal of $\iota^*\rho$ iff for every $\bar{\pi}_1$-vertical Chetaev vector field Z

$$J^1\gamma^* i_Z\, \mathrm{d}\iota^*\rho = 0. \tag{4.60}$$

Indeed, it is sufficient to note that every $\bar{\pi}_1$-vertical vector field can be expressed as a sum of vector fields with compact supports on appropriate pieces of X. Next, since any constraint $(n+1)$-form φ on Q reads $\varphi = \varphi^{(1)} \wedge \eta$, where $\varphi^{(1)} \in \mathcal{C}^0$ and η is an n-form on Q, we have $i_Z\varphi = i_Z(\varphi^{(1)} \wedge \eta) = -\varphi^{(1)} \wedge i_Z\eta$. The latter is, however, a constraint form, vanishing along $J^1\gamma$ (since it is contact). Thus, we have proved equivalence of (1) and (2).

Equivalence of (1), (5) and (6) is easily seen from the following expression of the constrained first variation formula

$$\int_\Omega J^1\gamma^*\mathcal{L}_Z\iota^*\rho = \int_\Omega J^2\gamma^* i_Z\bar{E}_\lambda + \int_\Omega J^1\gamma^* \mathrm{d}i_Z\rho. \tag{4.61}$$

Indeed, by analogous arguments as in the previous case, γ is an extremal of $\iota^*\rho$ iff for every $\bar{\pi}_1$-vertical Chetaev vector field Z

$$J^2\gamma^* i_Z\bar{E}_\lambda = 0. \tag{4.62}$$

Using the expression (4.47) for $\bar{E}_\lambda$, we obtain

$$0 = J^2\gamma^* i_Z\bar{E}_\lambda = J^2\gamma^*(\mathcal{E}^{\mathrm{c}}_s Z^s\omega_0 + \text{ a constraint form}) = J^2\gamma^*(\mathcal{E}^{\mathrm{c}}_s Z^s\omega_0),$$

where Z^s are components of a $\bar{\pi}_1$-vertical Chetaev vector field. Using the requirement that this condition has to be satisfied for every $\bar{\pi}_1$-vertical Chetaev field, and looking at the generators of the canonical distribution $\mathcal{C}$, we conclude that this is the case iff (6) holds. The latter is, however, a coordinate form of (5).

Let us show that (2) implies (4) (the converse is obvious). Let Z be an arbitrary Chetaev vector field. Then it can be locally splitted as $Z = Z^i\frac{\partial}{\partial x^i} + Z_{\mathrm{v}}$, where Z_{v} is $\bar{\pi}_1$-vertical. Assuming (2), we have for any constraint $(n+1)$-form φ (by similar arguments as above)

$$J^1\gamma^* i_Z(\mathrm{d}\iota^*\rho + \varphi) = J^1\gamma^*(Z^i i_{\partial/\partial x^i}\iota^*\mathrm{d}\rho) = J^2\gamma^*(Z^i i_{\partial/\partial x^i}\bar{E}_\lambda) = -J^2\gamma^*\big(\mathcal{E}^{\mathrm{c}}_s Z^i\bar{\omega}^s \wedge \omega_i\big) = 0,$$

since $\mathcal{E}^{\mathrm{c}}_s = 0$, due to the already proved equivalence of (2) and (6).

The equivalence of (2) and (3) is proved exactly by the same arguments.

The independence upon a choice of ρ is now trivial, being an immediate consequence of equivalence of the corresponding equations with (5) or (6). □

Note that the requirement that vector fields appearing in the intrinsic constrained Euler–Lagrange equations be Chetaev vector fields is essential. Indeed, given a vector field ζ on Q not belonging to the canonical distribution, we have

$$J^1\gamma^* i_\zeta \mathrm{d}\iota^*\rho = J^2\gamma^* i_\zeta\,\bar{p}_1\,\mathrm{d}\iota^*\rho = J^2\gamma^* i_\zeta\bar{E}_\lambda = J^2\gamma^*\big(\mathcal{E}^{\mathrm{c}}_s\zeta^s\omega_0 - \mathcal{E}^{\mathrm{c}}_s\zeta^i_0\bar{\omega}^s \wedge \omega_i + i_\zeta(\varphi^{(1)} \wedge \omega_0)\big),$$

where $\varphi^{(1)}$ is a constraint 1-form. If γ is an extremal, then $J^1\gamma^* i_\zeta \mathrm{d}\iota^*\rho = J^1\gamma^*(i_\zeta\varphi^{(1)} \cdot \omega_0) \neq 0$.

We have seen that given a Lagrangian λ on J^1Y, the constrained Euler–Lagrange equations do not depend upon a choice of a Lepage equivalent ρ of λ, i.e. they are *the same* for all Lepage equivalents ρ of λ. This is, however, important, since for different Lepage equivalents of λ, the Lagrangian λ on J^1Y gives rise to *different constrained Lagrangian systems* $\iota^*\rho$ on Q.

Using this result, the following definition can be stated: given a Lagrangian λ on J^1Y, a section γ of $\pi : Y \to X$ is called a *constrained extremal* of λ, if $\mathrm{Im}J^1\gamma \subset Q$, and

$$\int_\Omega J^1\gamma^*\mathcal{L}_Z\iota^*\rho = 0, \tag{4.63}$$

for (any) Lepage equivalent ρ of λ, and every 'fixed endpoints' variation Z over Ω, or, equivalently,

$$\int_\Omega J^1\gamma^*\mathcal{L}_Z\iota^*\Theta_\lambda = 0, \tag{4.64}$$

for every 'fixed endpoints' variation Z over Ω. γ is called a *constrained extremal of* λ if it is its constrained extremal on every piece Ω of X such that $\Omega \subset \mathrm{Dom}\gamma$.

Remark 4.18. Constrained Euler–Lagrange equations in item (6) of the above theorem are the *reduced nonholonomic equations* (so-called 'Chetaev equations without Lagrange multipliers'). In a different way these equations have been deduced in [15, 24] (mechanics) and [16, 20] (field theory).

Note that constrained Euler–Lagrange equations are determined by L, defined on the unconstrained evolution space, or by $k+1$ functions $\overline{L}$ and L_a, $1 \leqslant a \leqslant k$ (respectively, $\kappa + 1$ functions $\overline{L}$ and L_a^j, $(a, j) \notin \mathcal{J}$) on Q, where k (respectively κ) is the number of equations defining the constraint. The constrained Euler–Lagrange equations *cannot be determined by the restricted Lagrangian* $\bar{L}$ *alone*, unless $\psi^a = 0$ $\left(\psi^a_j = 0\right)$, i.e. *unless the constraint is semiholonomic*. In this sense, a *constrained Lagrangian is a (local) n-form*

$$\lambda_c = \bar{L}\,\mathrm{d}t + \sum_{a=1}^{k} L_a \varphi^a, \qquad \text{resp.} \quad \lambda_c = \bar{L}\omega_0 + \sum_{(a,j)\notin\mathcal{J}} L_a^j \varphi^a \wedge \omega_j, \tag{4.65}$$

that cannot be replaced by a single function on Q.

4.4. An alternative variational principle for simple nonholonomic and semiholonomic constraints

We have seen that the family of nonholonomic constraints contains a geometrically significant subfamily—*constraints that can be modelled by a distribution on* Y; according to [14] such constraints are called *simple nonholonomic constraints*. Recall that in mechanics this family is quite large and important, since it contains all nonholonomic constraints affine in velocities; on the other hand, in field theory it is only a certain subfamily of 'affine' constraints.

Below we shall present a modification of the nonholonomic variational principle suitable for description of this particular situation.

On Y consider a weakly horizontal distribution $\mathcal{D}$ of a constant corank $k < m$ (hence of rank $n + m - k$); recall that weak horizontality means that $\mathcal{D}$ has a vertical subdistribution of rank $m - k$ (sections of π are among admissible integral mappings) [13]. As mentioned earlier, $\mathcal{D}$ gives rise to a constraint $Q \subset J^1Y$ and the canonical distribution $\mathcal{C}$ on Q (equations for holonomic sections of $\bar{\pi}_1 : Q \to X$ are equations for integral sections of $\mathcal{D}$, i.e. *admissible sections of* π *are integral sections of* $\mathcal{D}$, and it holds that $\mathcal{C}$ projects onto $\mathcal{D}$).

Semiholonomic constraints. First, assume that $\mathcal{D}$ is *completely integrable*, i.e., the nonholonomic constraint is *semiholonomic*. In this case $\mathcal{D}$ is spanned by π-projectable vector fields ξ on Y such that $J^1_c\xi \in \mathcal{C}$. Moreover, the vector fields $J^1_c\xi$ are symmetries of the induced contact ideal on Q, meaning that the Lie derivative preserves decomposition of forms into contact components. All this means that the variational principle does not much differ from the unconstrained and holonomic one:

- *Integral sections* γ *of the distribution* $\mathcal{D}$ *such that* $\mathrm{Im}\, J^1\gamma \cap Q \neq \emptyset$ *are admissible sections*, since their prolongations satisfy equations of Q. Indeed, if γ is an integral section of $\mathcal{D}$ then γ is an integral section of a vector field $\xi \in \mathcal{D}$, and hence $J^1\gamma$ is an integral section of $J^1\xi$. However, $J^1\xi$ along Q is tangent to Q and equal to $J^1_c\xi$, so that if for some $x \in X$, $J^1\gamma(x) \in Q$ then $\mathrm{Im} J^1\gamma \subset Q$, meaning that γ is an admissible section of π.
- Admissible variations in Y are π-projectable vector fields belonging to $\mathcal{D}$.

- *Admissible variations in Y obey the principle of virtual displacements*. Indeed, deformations of an admissible section γ of π by projectable $\xi \in \mathcal{D}$ induce deformations of $J^1\gamma$ by $J^1_c\xi \in \mathcal{C}$ that all are holonomic sections of Q: $J^1\gamma_u = (J^1\gamma)_u$, $\forall u$.
- We can *restrict the constrained action to the subset of admissible sections of π*. The variation of the restricted action is well defined and equal to the restriction of the variation of the constrained action.
- In the integrand of both the action and the variation of the action, only *the horizontal part*, i.e. the *restricted* (to Q) *Lagrangian*, $\iota^*\lambda = \overline{\lambda}$, is essential.

Summarizing, given a Lagrangian λ on J^1Y, we have the *constrained action function*

$$\mathcal{S}_{\Omega,Q}(\pi) \ni \gamma \to \int_\Omega J^1\gamma^*\iota^*\rho = \int_\Omega J^1\gamma^*\overline{\lambda} \in \mathbb{R}, \tag{4.66}$$

and for *every π-projectable vector field* $\xi \in \mathcal{D}$ the first variation of the constrained action induced by ξ is computed in a standard way; this computation gives

$$\mathcal{S}_{\Omega,Q}(\pi) \ni \gamma \to \int_\Omega J^1\gamma^*\mathcal{L}_{J^1_c\xi}\iota^*\rho = \int_\Omega J^1\gamma^*\mathcal{L}_{J^1_c\xi}\overline{\lambda} \in \mathbb{R}. \tag{4.67}$$

(cf with (4.36)). Now, the *semiholonomic first variation formula* takes one of the following equivalent forms:

$$\int_\Omega J^1\gamma^*\mathcal{L}_{J^1_c\xi}\overline{\lambda} = \int_\Omega J^1\gamma^* i_{J^1_c\xi}\,\mathrm{d}\iota^*\rho + \int_{\partial\Omega} J^1\gamma^* i_{J^1_c\xi}\iota^*\rho, \tag{4.68}$$

$$\int_\Omega J^1\gamma^*\mathcal{L}_{J^1_c\xi}\overline{\lambda} = \int_\Omega J^1\gamma^* i_{J^1_c\xi}\,\mathrm{d}\Theta_{\overline{\lambda}} + \int_{\partial\Omega} J^1\gamma^* i_{J^1_c\xi}\Theta_{\overline{\lambda}}. \tag{4.69}$$

We can see that

$$\bar{E}_\lambda = \iota^* p_1\,\mathrm{d}\Theta_{\overline{\lambda}}, \tag{4.70}$$

and semiholonomic Euler–Lagrange equations are equations for integral sections of $\mathcal{D}$, such that

$$\begin{aligned} &\dim X = 1 \qquad \frac{\partial_c \bar{L}}{\partial q^s} - \frac{\mathrm{d}_c}{\mathrm{d}t}\frac{\partial \bar{L}}{\partial \dot{q}^s} = 0, \\ &\dim X > 1 \qquad \frac{\partial_c \bar{L}}{\partial y^s} - \frac{\mathrm{d}_c}{\mathrm{d}x^j}\frac{\partial \bar{L}}{\partial y^s_j} = 0. \end{aligned} \tag{4.71}$$

Simple nonholonomic constraints. The next case to be discussed is such that the distribution $\mathcal{D}$ *is not integrable*. Now, $\mathcal{D}$ is spanned by projectable vector fields ξ on Y however, at the points of Q, $J^1\xi$ need not be tangent to Q ($J^1_c\xi$ need not be defined). This means that $\xi \in \mathcal{D}$ need not have a counterpart Z in $\mathcal{C}$ that would be a symmetry of the induced contact ideal on Q, and consequently, the Lie derivative along Z does not preserve decomposition of forms into contact components. We conclude that:

- *Integral sections of the distribution* $\mathcal{D}$ are projections of admissible sections of Q.
- Admissible variations in Y are π-projectable vector fields belonging to $\mathcal{D}$.
- *Admissible variations in Y do not obey the principle of virtual displacements*: deformations of an admissible section γ of π by projectable $\xi \in \mathcal{D}$ induce deformations of $J^1\gamma$ by $Z \in \mathcal{C}$ such that ξ is a projection of Z; however, the deformations *need not be holonomic* sections of Q ($J^1\gamma_u \neq (J^1\gamma)_u$); the $J^1\gamma_u$ even need not be *sections* of Q ($\mathrm{Im} J^1\gamma_u$ need not be a subset of Q). Instead, we have a *nonholonomic principle of virtual displacements* that for simple nonholonomic constraints takes the form

$$(J^1\gamma)_u = \delta_u, \qquad \gamma_u = \bar{\pi}_{1,0}\delta_u \tag{4.72}$$

(for all $u \in (-\epsilon, \epsilon)$), where $\{\delta_u\}$ is a deformation induced by a Chetaev vector field, and $\{\gamma_u\}$ by its projection onto Y. It can be briefly stated as follows: 'every admissible variation of derivation induces a variation in the space of events'.

- We *cannot restrict the constrained action to a subset of admissible sections of* π, since deformations $(-\epsilon, \epsilon) \ni u \to \phi_u J^1\gamma\phi_{0u}$ need not belong to the subset of holonomic sections of the projection $\bar{\pi}_1$, so that the composition

$$u \to \delta_u = \phi_u J^1\gamma\phi_{0u}^{-1} \to \int_{\phi_{0u}(\Omega)} \delta_u^* \iota^* \rho \tag{4.73}$$

would not be defined.

- In the integrand of the action and the variation of the action one cannot forget about the contact part (the restricted Lagrangian $\overline{\lambda}$ is not sufficient).

Summarizing, there are no significant simplifications available, and the situation is similar to the general case: the constrained action is

$$\mathcal{S}_\Omega(\bar{\pi}_1) \ni \delta \to \int_\Omega \delta^* \iota^* \rho \in \mathbb{R}, \tag{4.74}$$

its variation induced by a $\bar{\pi}_1$ and $\bar{\pi}_{1,0}$-projectable Chetaev vecor field Z takes the form

$$\mathcal{S}_\Omega(\bar{\pi}_1) \ni \delta \to \left(\frac{\mathrm{d}}{\mathrm{d}u} \int_{\phi_{0u}(\Omega)} \delta_u^* \iota^* \rho \right)_{u=0} = \int_\Omega \delta^* \mathcal{L}_Z \iota^* \rho \in \mathbb{R}, \tag{4.75}$$

and can be restricted to constrained sections of the fibred manifold $\pi : Y \to X$ (admissible sections of π) to become

$$\mathcal{S}_{\Omega,Q}(\pi) \ni \gamma \to \int_\Omega J^1\gamma^* \mathcal{L}_Z \iota^* \rho \in \mathbb{R}, \tag{4.76}$$

as in the general case. Note that, however, in this case *sections belonging to* $\mathcal{S}_{\Omega,Q}(\pi)$ *are integral sections of* $\mathcal{D}$.

4.5. A general nonholonomic first variation formula

The second (and more general) possibility is to consider a variational principle such that not only variations, but also *the system to be extremized is defined on the nonholonomic constraint submanifold*, without any reference to the 'ambient space' J^1Y. It is apparent that we need to say what such an 'inside Lagrangian system' should be. Naturally we do require that if an 'inside system' arises from a Lagrangian (or better from its Lepage equivalent ρ) on J^1Y as $\iota^*\rho$, then the results are reduced to the previous ones. Accounting all this, we can deduce that the integrand of the action of an 'inside nonholonomic variational principle' *cannot be* a horizontal form, or even a *function* on Q (a 'Lagrangian'). It must be an n-form with similar properties as *Lepage forms* in the unconstrained situation—for a constrained action we need the concept of a *constrained Lepage form* [21][6].

Definition 4.19. *Let $Q \subset J^1Y$ be a nonholonomic constraint. We call a differential n-form $\bar{\rho}$ on Q constrained Lepage form if $\bar{p}_1\, \mathrm{d}\bar{\rho}$ is $\bar{\pi}_{2,0}$-horizontal.*

[6] Indeed, there is no reason to assume *a priori* that the action of a differential form and of its horizontal part should be the same; and as we have seen, this really is *not* the case, unless the constraint is semiholonomic. Moreover, for the first variation formula we *do need* the Lepage property assuring that the action integral is related to a *dynamical form*.

First-order constrained Lepage forms take the following coordinate expression:

$$\text{if } \dim X = 1: \qquad \bar{\rho} = \overline{L}\,\mathrm{d}t + \sum_{l=1}^{m-k} \frac{\partial \overline{L}}{\partial \dot{q}^l} \bar{\omega}^l + \sum_{a=1}^{k} L_a \varphi^a,$$

$$\text{if } \dim X > 1: \qquad \bar{\rho} = \overline{L}\omega_0 + \sum_{J=(\sigma,i)\in\mathcal{J}} \left(\frac{\partial \overline{L}}{\partial z^J} - \sum_{(\nu,p)\notin\mathcal{J}} L_\nu^p \frac{\partial g_p^\nu}{\partial z^J} \right) \bar{\omega}^\sigma \wedge \omega_i + \sum_{(\nu,p)\notin\mathcal{J}} L_\nu^p \bar{\omega}^\nu \wedge \omega_p + \mu, \tag{4.77}$$

where μ is an arbitrary at least 2-contact n-form on Q.

Let $Q \subset J^1Y$ be a nonholonomic constraint, $\mathcal{C}$ the canonical distribution on Q.

Definition 4.20. *By a Lagrangian system on a nonholonomic constraint Q we shall mean a constrained Lepage form $\bar{\rho}$ on Q.*

From now on, the setting and procedure to obtain constrained action, variation of the action, first variation formula and Euler–Lagrange equations is the same as in the case of constrained ambient Lagrangian systems: only $\iota^*\rho$ should be replaced by $\bar{\rho}$.

Note that the action, first variation and Euler–Lagrange equations cannot be obtained provided only *one function* on Q (a 'constrained Lagrange function') is given: indeed, they depend upon a choice of $k+1$ functions $\overline{L}$, L_a, $1 \leqslant a \leqslant k$, if $\dim X = 1$, respectively $\kappa + 1$ functions $\overline{L}$, L_ν^p, where $(\nu, p) \notin \mathcal{J}$, if $\dim X > 1$.

It is worth noting that in this case *a corresponding unconstrained system need not exist*, and *if it exists it need not be Lagrangian*. We refer the reader to [18] for more details on non-Lagrangian systems on J^1Y that become (constrained) Lagrangian systems if subject to an appropriate nonholonomic constraint. For illustration, we give an example of such a system below.

5. Examples of nonholonomic variational systems

In this section we give examples of mechanical systems subject to nonholonomic constraints, that are 'constraint-variational', however, cannot be obtained via the traditional variational procedure. As we have seen, this concerns 'true' nonholonomic systems, i.e. systems that are not semiholonomic (the constraints are nonintegrable).

In the examples below, we consider a nonholonomic constraint Q of codimension 1, given by a single equation, in normal form denoted as

$$\dot{q}^m = g(t, q^\sigma, \dot{q}^l), \tag{5.1}$$

where $1 \leqslant \sigma \leqslant m$ and $1 \leqslant l \leqslant m-1$. Hence, we write g instead of g^1, and similarly, φ instead of φ^1 for the corresponding constraint form; in this notation,

$$\varphi = -\sum_{l=0}^{m-1} \frac{\partial g}{\partial \dot{q}^l} (\mathrm{d}q^l - \dot{q}^l\,\mathrm{d}t) + \mathrm{d}q^m - g\,\mathrm{d}t. \tag{5.2}$$

Recall that main points characterizing the nonholonomic variational procedure (making it different from the unconstrained, holonomic and semiholonomic ones) are the following:

- The constrained Lagrangian system, constrained variations, etc are defined on the constraint manifold Q that represents the *evolution space for the constrained system*.

- In place of a 'constraint Lagrangian' we shall have a 1-form

$$\lambda_c = \overline{L}\,\mathrm{d}t + L_1\varphi, \tag{5.3}$$

i.e. the constrained system will locally be represented by *two functions*, $\overline{L}$, L_1 (and as we know, *cannot* be reduced to a single function) on Q.[7]
- Constrained variations are projectable (onto the base) *Chetaev vector fields* living on Q, that *are not prolongations* of vector fields on the 'space of events' Y.

The examples are chosen in such a way that

- the first nonholonomic system arises from a Lagrangian system, subject to a constraint *linear in velocities*, i.e. representable as a *nonintegrable distribution on Y*;
- the second nonholonomic system arises from a Lagrangian system, subject to a constraint *quadratic in velocities*, i.e. *not representable as a distribution on Y*;
- the third example illustrates the most general possibility that has no counterpart in 'constrained *Lagrangian* mechanics'. Namely, the original (unconstrained) mechanical system *is not Lagrangian;* however, *the arising nonholonomic system is variational as a constrained system.*

5.1. *Example of a Lagrangian system subject to a linear nonintegrable constraint*

We shall consider a free particle of mass $m = 1$ moving in $\mathbb{R}^2$ along a curve the angular coefficient of which is proportional to the time that has passed from the beginning of the motion [28]. We have the fibred manifold $\pi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ with canonical coordinates (t, x, y), and the first jet prolongation $J^1(\mathbb{R}\times\mathbb{R}^2) = \mathbb{R}\times T\mathbb{R}^2$ with coordinates $(t, x, y, \dot{x}, \dot{y})$. The unconstrained system is given by the Lagrangian

$$\lambda = L\,\mathrm{d}t = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2)\,\mathrm{d}t \tag{5.4}$$

on $\mathbb{R}\times T\mathbb{R}^2$, and the constraint $Q \subset \mathbb{R}\times T\mathbb{R}^2$ (the *evolution space* of the constrained system) is given by equation

$$ct\dot{x} - \dot{y} = 0, \tag{5.5}$$

where c is a constant. Putting

$$\dot{y} = g(t, x, y, \dot{x}) \equiv ct\dot{x}, \tag{5.6}$$

we get the constraint equation in normal form. Now, $(t, x, y, \dot{x})$ are adapted coordinates on Q. In these coordinates, the constraint form φ on Q, annihilating the canonical distribution $\mathcal{C}$, reads

$$\varphi = -\frac{\partial g}{\partial \dot{x}}(\mathrm{d}x - \dot{x}\,\mathrm{d}t) + \mathrm{d}y - g\,\mathrm{d}t = -ct\,\mathrm{d}x + \mathrm{d}y. \tag{5.7}$$

The canonical distribution $\mathcal{C}$ would be completely integrable if the ideal generated by the 1-form φ would be closed, i.e., if $\mathrm{d}\varphi = -c\,\mathrm{d}t\wedge\mathrm{d}x = \varphi\wedge\eta$ for some 1-form η on Q. However, an easy computation shows that no such η exists. This means that the constraint ideal is not closed, i.e. the canonical distribution $\mathcal{C}$ *is not completely integrable*, i.e. the constraint Q is not semiholonomic.

Let us compute admissible variations: the canonical distribution is spanned by the following three vector fields:

$$\frac{\partial_c}{\partial t} = \frac{\partial}{\partial t} + \left(g - \frac{\partial g}{\partial \dot{x}}\dot{x}\right)\frac{\partial}{\partial y} = \frac{\partial}{\partial t}, \qquad \frac{\partial_c}{\partial x} = \frac{\partial}{\partial x} + \frac{\partial g}{\partial \dot{x}}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} + ct\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial \dot{x}}, \tag{5.8}$$

[7] If we considered k constraints, the 'constraint Lagrangian' λ_c would contain $k + 1$ 'Lagrangian functions'. In a more geometrical setting, instead of a Lepage form (in mechanics a Cartan form Θ_λ) in the integrand of the action there appears a *constraint Lepage form* that cannot be determined by a horizontal form (a Lagrangian) on Q.

so that we get (projectable onto $\mathbb{R}$) Chetaev vector fields = *admissible variations* of the form

$$Z = Z_0 \frac{\partial}{\partial t} + Z_1 \left(\frac{\partial}{\partial x} + ct \frac{\partial}{\partial y} \right) + Z_2 \frac{\partial}{\partial \dot{x}}, \tag{5.9}$$

where Z_0, Z_1, Z_2 are arbitrary functions on Q, $Z_0 = Z_0(t)$.

In order to write the constrained action of the Lagrangian λ we shall need the Cartan form

$$\Theta_\lambda = L\,\mathrm{d}t + \frac{\partial L}{\partial \dot{x}}(\mathrm{d}x - \dot{x}\,\mathrm{d}t) + \frac{\partial L}{\partial \dot{y}}(\mathrm{d}y - \dot{y}\,\mathrm{d}t) = -\frac{1}{2}(\dot{x}^2 + \dot{y}^2)\,\mathrm{d}t + \dot{x}\,\mathrm{d}x + \dot{y}\,\mathrm{d}y. \tag{5.10}$$

The *constrained system* is the 1-form $\iota^*\Theta_\lambda$. Locally it can be represented by a 'constraint Lagrangian'

$$\lambda_\mathrm{c} = \iota^*\lambda + \left(\frac{\partial L}{\partial \dot{y}} \circ \iota \right) \varphi = \frac{1}{2}\dot{x}^2(1 + c^2t^2)\,\mathrm{d}t + ct\dot{x}\varphi. \tag{5.11}$$

Now, we have the *constrained action*

$$\mathcal{S}_\Omega(\bar{\pi}_1) \ni \delta \to \int_\Omega \delta^*\iota^*\Theta_\lambda = \int_\Omega \delta^* \left(\dot{x}\,\mathrm{d}x + ct\dot{x}\,\mathrm{d}y - \frac{1}{2}\dot{x}^2(1 + c^2t^2)\,\mathrm{d}t \right) \in \mathbb{R}, \tag{5.12}$$

and, given a Chetaev vector field Z, the *variation of the constrained action* induced by Z

$$\mathcal{S}_\Omega(\bar{\pi}_1) \ni \delta \to \int_\Omega \delta^*\mathcal{L}_Z\iota^*\Theta_\lambda = \int_\Omega \delta^*\mathcal{L}_Z \left(\dot{x}\,\mathrm{d}x + ct\dot{x}\,\mathrm{d}y - \frac{1}{2}\dot{x}^2(1 + c^2t^2)\,\mathrm{d}t \right) \in \mathbb{R}. \tag{5.13}$$

Restricting the domain of definition to holonomic sections, $\delta = J^1\gamma$, we finally obtain

$$\begin{aligned}\mathcal{S}_{\Omega,Q}(\pi) \ni \gamma \to \int_\Omega J^1\gamma^*\mathcal{L}_Z\iota^*\Theta_\lambda &= \int_\Omega J^1\gamma^*\mathcal{L}_Z \left(\dot{x}\,\mathrm{d}x + ct\dot{x}\,\mathrm{d}y - \frac{1}{2}\dot{x}^2(1 + c^2t^2)\,\mathrm{d}t \right)\\ &= \int_\Omega J^1\gamma^* i_Z d \left(\dot{x}\,\mathrm{d}x + ct\dot{x}\,\mathrm{d}y - \frac{1}{2}\dot{x}^2(1 + c^2t^2)\,\mathrm{d}t \right)\\ &\quad + \int_{\partial\Omega} \left(\dot{x}(1 + c^2t^2)\big(Z_1 - \frac{1}{2}\dot{x}Z_0\big) \right) \circ J^1\gamma.\end{aligned} \tag{5.14}$$

To compute the constrained Euler–Lagrange equation it is sufficient to consider *vertical* Chetaev vector fields. We obtain

$$\begin{aligned}&\bar{h} i_Z d \left(\dot{x}\,\mathrm{d}x + ct\dot{x}\,\mathrm{d}y - \tfrac{1}{2}\dot{x}^2(1 + c^2t^2)\,\mathrm{d}t \right)\\ &\quad = \bar{h} i_Z(\mathrm{d}\dot{x} \wedge \mathrm{d}x + ct\,\mathrm{d}\dot{x} \wedge \mathrm{d}y + c\dot{x}\,\mathrm{d}t \wedge \mathrm{d}y - \dot{x}(1 + c^2t^2)\,\mathrm{d}\dot{x} \wedge \mathrm{d}t)\\ &\quad = \bar{h}(Z_2\,\mathrm{d}x - Z_1\,\mathrm{d}\dot{x} + ctZ_2\,\mathrm{d}y - c^2t^2Z_1\,\mathrm{d}\dot{x} - c^2t\dot{x}Z_1\,\mathrm{d}t - \dot{x}(1 + c^2t^2)Z_2\,\mathrm{d}t)\\ &\quad = -(\ddot{x}(1 + c^2t^2) + c^2t\dot{x})Z_1\,\mathrm{d}t,\end{aligned} \tag{5.15}$$

giving the *constrained Euler–Lagrange equation*

$$\ddot{x}(1 + c^2t^2) + c^2t\dot{x} = 0. \tag{5.16}$$

It is worth noting that the canonical distribution $\mathcal{C}$ is, indeed, projectable onto a nonintegrable distribution $\mathcal{D}$ of rank 2 on the evolution space $\mathbb{R} \times \mathbb{R}^2$: $\mathcal{D}$ is annihilated by following 1-form $-ct\,\mathrm{d}x + \mathrm{d}y$ on $\mathbb{R} \times \mathbb{R}^2$, or, equivalently, spanned by vector fields

$$\xi = \xi_0 \frac{\partial}{\partial t} + \xi_1 \left(\frac{\partial}{\partial x} + ct \frac{\partial}{\partial y} \right). \tag{5.17}$$

As we have seen, vector fields belonging to $\mathcal{D}$ such that $\xi_0 = \xi_0(t)$ *induce nonholonomic variations of paths in* $\mathbb{R} \times \mathbb{R}^2$. We can also easily check that these vector fields do not admit prolongation to Q. Computing the prolongation condition we obtain

$$\frac{\mathrm{d}'_c(\xi_1 ct)}{\mathrm{d}t} - \frac{\partial g}{\partial y}ct = \frac{\partial g}{\partial t}\xi_0 + \frac{\partial g}{\partial x}\xi_1 + \frac{\partial g}{\partial \dot{x}}\frac{\mathrm{d}'_c\xi_1}{\mathrm{d}t} + \left(g - \frac{\partial g}{\partial \dot{x}}\dot{x} \right) \frac{\mathrm{d}'_c\xi_1}{\mathrm{d}t}, \tag{5.18}$$

i.e.,

$$\xi_1 c = c\dot{x}\xi_0. \tag{5.19}$$

This is a linear equation in $\dot{x}$, having the only solution $\xi_1 = 0$, $\xi_0 = 0$. Hence, for no nonzero vector field $\xi \in \mathcal{D}$, the vector field $J^1\xi$ along Q is tangent to Q, meaning that *prolongations* of vector fields belonging to the distribution $\mathcal{D}$ *do not induce variations of curves in the evolution space* Q.

5.2. *Example of a Lagrangian system subject to a nonlinear constraint*

The second example concerns a relativistic particle considered as a nonholonomic system [17], illustrating a Lagrangian system subject to a nonlinear constraint. We have seen that such a constraint cannot be represented by a distribution on the evolution space; however, is modelled as a submanifold Q in the evolution space of the original (unconstrained) system, with the canonical distribution $\mathcal{C}$ on Q.

We consider a fibred manifold $\pi : \mathbb{R}\times\mathbb{R}^4 \to \mathbb{R}$ and its jet prolongation $\pi_1 : \mathbb{R}\times T\mathbb{R}^4 \to \mathbb{R}$, with global fibred coordinates denoted by $(s, q^\sigma, \dot{q}^\sigma)$, $1 \leqslant \sigma \leqslant 4$, where $\mathbb{R}^4$ is endowed with the Minkowski metric $-(\mathrm{d}q^1)^2 - (\mathrm{d}q^2)^2 - (\mathrm{d}q^3)^2 + (\mathrm{d}q^4)^2$. Let $\lambda = L\,\mathrm{d}s$ be a Lagrangian on $\mathbb{R}\times T\mathbb{R}^4$,

$$L = -\frac{1}{2}m_0\left((\dot{q}^4)^2 - \sum_{p=1}^{3}(\dot{q}^p)^2\right) + \phi_\sigma\dot{q}^\sigma - \psi, \tag{5.20}$$

where $m_0 > 0$ is a constant (the rest mass of the particle), and $(\phi_\sigma) = (\vec{A}, -V)$ and ψ are functions on $\mathbb{R}^4$, representing a 4-potential and a scalar potential, respectively. Let a nonholonomic constraint Q in $\mathbb{R}\times T\mathbb{R}^4$ be given by the equation

$$(\dot{q}^4)^2 - \sum_{p=1}^{3}(\dot{q}^p)^2 = 1, \tag{5.21}$$

expressing the relativistic condition on the 4-velocity. Assume $\dot{q}^4 > 0$, and take for the constrained system the evolution space

$$Q_+ \subset Q \subset \mathbb{R}\times T\mathbb{R}^4 : \qquad \dot{q}^4 = g(s, q^\sigma, \dot{q}^1, \dot{q}^2, \dot{q}^3) = \sqrt{1 + \sum_{p=1}^{3}(\dot{q}^p)^2}, \tag{5.22}$$

with adapted coordinates $(s, q^\sigma, \dot{q}^l)$, $1 \leqslant l \leqslant 3$.

The nonholonomic system on Q_+ is given by the 1-form $\iota^*\Theta_\lambda$, where Θ_λ is the Cartan form of the Lagrangian λ,

$$\begin{aligned}\Theta_\lambda = L\,\mathrm{d}s + \frac{\partial L}{\partial\dot{q}^\sigma}\omega^\sigma &= -\left(\frac{1}{2}m_0\left((\dot{q}^4)^2 - \sum_{p=1}^{3}(\dot{q}^p)^2\right) - A_p\dot{q}^p + V\dot{q}^4 + \psi\right)\mathrm{d}s \\ &\quad + \sum_{p=0}^{3}(m_0\dot{q}^p + A_p)(\mathrm{d}q^p - \dot{q}^p\,\mathrm{d}s) - (m_0\dot{q}^4 + V)(\mathrm{d}q^4 - \dot{q}^4\,\mathrm{d}s),\end{aligned} \tag{5.23}$$

and ι is the canonical embedding of Q_+ into $\mathbb{R}\times T\mathbb{R}^4$, $\dot{q}^4 \circ \iota = g$.

It is convenient to consider on $\mathbb{R}\times T\mathbb{R}^4 - \{\dot{q}^4 = 0\}$ other coordinates $(s, q^l, t, v^l, \dot{q}^4)$, $1 \leqslant l \leqslant 3$, better adapted to a three-dimensional observer, defined as follows:

$$t = q^4, \qquad v^l = \frac{\dot{q}^l}{\dot{q}^4}. \tag{5.24}$$

In these coordinates, the constraint Q_+ is given by the equation

$$\dot{q}^4 = \frac{1}{\sqrt{1-v^2}}, \tag{5.25}$$

and the *constrained Lagrangian system* takes the form

$$\iota^*\Theta_\lambda = \left(\frac{1}{2}m_0 - \psi\right)ds + \sum_{l=1}^{3}\left(\frac{m_0 v^l}{\sqrt{1-v^2}} + A_l\right)\mathrm{d}q^l - \left(\frac{m_0}{\sqrt{1-v^2}} + V\right)\mathrm{d}t. \tag{5.26}$$

We shall find *admissible variations*. The canonical distribution $\mathcal{C}$ on Q_+ is annihilated by an 1-form

$$\begin{aligned}\varphi &= -\sum_{l=0}^{3}\frac{\dot{q}^l}{\sqrt{1+\sum_{p=1}^{3}(\dot{q}^p)^2}}(\mathrm{d}q^l - \dot{q}^l\,\mathrm{d}s) + \left(\mathrm{d}q^4 - \sqrt{1+\sum_{p=1}^{3}(\dot{q}^p)^2\,\mathrm{d}s}\right)\\ &= \mathrm{d}t - \sum_{l=0}^{3}v^l\,\mathrm{d}q^l - \sqrt{1-v^2}\,\mathrm{d}s,\end{aligned} \tag{5.27}$$

or, equivalently, spanned by seven vector fields

$$\begin{aligned}\frac{\partial_c}{\partial s} &= \frac{\partial}{\partial s} + \left(g - \frac{\partial g}{\partial \dot{q}^l}\dot{q}^l\right)\frac{\partial}{\partial q^4} = \frac{\partial}{\partial s} + \sqrt{1-v^2}\frac{\partial}{\partial t},\\ \frac{\partial_c}{\partial q^l} &= \frac{\partial}{\partial q^l} + \frac{\partial g}{\partial \dot{q}^l}\frac{\partial}{\partial q^4} = \frac{\partial}{\partial q^l} + v^l\frac{\partial}{\partial t},\\ \frac{\partial}{\partial \dot{q}^l} &= \sqrt{1-v^2}\frac{\partial}{\partial v^l}.\end{aligned} \tag{5.28}$$

Let us write the constrained variational principle. The *constrained action* is

$$\begin{aligned}\mathcal{S}_\Omega(\bar{\pi}_1) \ni \delta \to &\int_\Omega \delta^*\iota^*\Theta_\lambda\\ &= \int_\Omega \delta^*\left(\left(\frac{1}{2}m_0 - \psi\right)\mathrm{d}s + \sum_{l=1}^{3}\left(\frac{m_0 v^l}{\sqrt{1-v^2}} + A_l\right)\mathrm{d}q^l\right.\\ &\quad\left. - \left(\frac{m_0}{\sqrt{1-v^2}} + V\right)\mathrm{d}t\right) \in \mathbb{R}.\end{aligned} \tag{5.29}$$

For a fixed Chetaev vector field $Z \in \mathcal{C}$, the *variation of the constrained action* induced by Z is

$$\begin{aligned}\mathcal{S}_\Omega(\bar{\pi}_1) \ni \delta \to &\int_\Omega \delta^*\mathcal{L}_Z\iota^*\Theta_\lambda\\ &= \int_\Omega \delta^*\mathcal{L}_Z\left(\left(\frac{1}{2}m_0 - \psi\right)\mathrm{d}s + \sum_{l=1}^{3}\left(\frac{m_0 v^l}{\sqrt{1-v^2}} + A_l\right)\mathrm{d}q^l\right.\\ &\quad\left. - \left(\frac{m_0}{\sqrt{1-v^2}} + V\right)\mathrm{d}t\right) \in \mathbb{R}.\end{aligned} \tag{5.30}$$

Restricting the domain of definition to holonomic sections, $\delta = J^1\gamma$, and putting

$$\begin{aligned}Z &= Z^0\frac{\partial_c}{\partial s} + Z^l\frac{\partial_c}{\partial q^l} + \hat{Z}^l\frac{\partial}{\partial \dot{q}^l}\\ &= Z^0\frac{\partial}{\partial s} + \sum_l Z^l\frac{\partial}{\partial q^l} + \left(\sum_l Z^l v^l + Z^0\sqrt{1-v^2}\right)\frac{\partial}{\partial t} + \sum_l \tilde{Z}^l\sqrt{1-v^2}\frac{\partial}{\partial v^l},\end{aligned} \tag{5.31}$$

we get the first variation of the constrained action in the form

$$\mathcal{S}_{\Omega,Q}(\pi) \ni \gamma \to \int_\Omega J^1\gamma^* \mathcal{L}_Z \iota^* \Theta_\lambda$$
$$= \int_\Omega J^1\gamma^* \mathcal{L}_Z \left(\left(\frac{1}{2}m_0 - \psi \right) \mathrm{d}s + \sum_{l=1}^{3} \left(\frac{m_0 v^l}{\sqrt{1-v^2}} + A_l \right) \mathrm{d}q^l - \left(\frac{m_0}{\sqrt{1-v^2}} + V \right) \mathrm{d}t \right)$$
$$= \int_\Omega J^1\gamma^* i_Z\, \mathrm{d} \left(\left(\frac{1}{2}m_0 - \psi \right) \mathrm{d}s + \sum_{l=1}^{3} \left(\frac{m_0 v^l}{\sqrt{1-v^2}} + A_l \right) \mathrm{d}q^l - \left(\frac{m_0}{\sqrt{1-v^2}} + V \right) \mathrm{d}t \right)$$
$$+ \int_{\partial\Omega} \left(\sum_l (A_l - V v^l) Z^l - \left(\frac{1}{2}m_0 + \psi + V\sqrt{1-v^2} \right) Z^0 \right) \circ J^1\gamma. \tag{5.32}$$

Taking for simplicity vertical Chetaev vector fields ($Z^0 = 0$), we obtain the *constrained Euler–Lagrange equations* by a straightforward computation. Similarly as in [17] we can write them as equations for sections $\gamma(s) = (s, t(s), q^l(t(s))$ in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0 \vec{v}}{\sqrt{1-v^2}} \right) = \vec{v} \times \mathrm{rot}\,\vec{A} - \frac{\partial \vec{A}}{\partial t} - \mathrm{grad}\, V - \sqrt{1-v^2}\,\mathrm{grad}\,\psi - \frac{\vec{v}}{\sqrt{1-v^2}} \frac{\mathrm{d}\psi}{\mathrm{d}t}. \tag{5.33}$$

Finally, note that the vector fields $\partial_c/\partial s, \partial_c/\partial q^l \in \mathcal{C}$ are *not projectable* onto $\mathbb{R} \times \mathbb{R}^4$, i.e., the distribution $\mathcal{C}$ on Q_+ does not have a counterpart on $\mathbb{R} \times \mathbb{R}^4$. This means that nonholonomic deformations of admissible sections $J^1\gamma$ passing in the evolution space Q_+ induce deformations of sections γ of π (projections of the admissible sections) that *are not induced by vector fields on* $\mathbb{R} \times \mathbb{R}^4$.

5.3. Example of a constrained non-Lagrangian system

Our last example is an illustration of a mechanical system that, if unconstrained, is not Lagrangian; however, under a nonholonomic constraint it turns into a constraint Lagrangian in our sense.

Following [5], let us consider equations of motion

$$m\ddot{x} + \beta\dot{x} - mG = 0, \qquad m\ddot{y} + \beta\dot{y} = 0, \tag{5.34}$$

describing a particle moving with friction in a gravitational field, and subject to the following nonholonomic constraint (representing conservation of the mechanical energy)

$$\tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mGx = c, \tag{5.35}$$

where c is a constant. The evolution space of the constrained system is the manifold $Q \subset \mathbb{R} \times T\mathbb{R}^2$, defined by the above equation. On Q_+, where

$$\dot{y} = g \equiv \sqrt{\frac{2c}{m} + 2Gx - \dot{x}^2} > 0, \tag{5.36}$$

we obtain the constraint equation of motion

$$m\ddot{x}\left(1 + \frac{\dot{x}^2}{g^2}\right) - mG\left(1 + \frac{\dot{x}^2}{g^2}\right) = 0. \tag{5.37}$$

This equation is variational: it comes from the nonholonomic variational principle

$$\mathcal{S}_\Omega(\bar{\pi}_1) \ni \delta \to \int_\Omega \delta^* \bar{\rho} \in \mathbb{R}, \tag{5.38}$$

where the *constrained Lepage form* reads as follows:

$$\bar{\rho} = -2mGx\,\mathrm{d}t - \left(m\sqrt{\frac{2c}{m} + 2Gx - \dot{x}^2}\right)\varphi, \tag{5.39}$$

where φ is an 1-form, annihilating the canonical distribution $\mathcal{C}$,

$$\varphi = -\frac{\partial g}{\partial \dot{x}}(\mathrm{d}x - \dot{x}\,\mathrm{d}t) + \mathrm{d}y - g\,\mathrm{d}t = \mathrm{d}y + \frac{\dot{x}}{g}\,\mathrm{d}x - \left(\frac{\dot{x}^2}{g} + g\right)\,\mathrm{d}t. \tag{5.40}$$

Substituting into $\bar{\rho}$ we obtain the *constrained action*

$$\mathcal{S}_\Omega(\bar{\pi}_1) \ni \delta \to \int_\Omega \delta^* \left(2c\,\mathrm{d}t - m\dot{x}\,\mathrm{d}x - m\sqrt{\frac{2c}{m} + 2Gx - \dot{x}^2}\,\mathrm{d}y\right) \in \mathbb{R}. \tag{5.41}$$

Let us check that this constrained action provides the desired equation of motion.

Admissible variations (Chetaev vector fields) take the form

$$Z = Z_0\frac{\partial_c}{\partial t} + Z_1\frac{\partial_c}{\partial x} + Z_2\frac{\partial}{\partial \dot{x}} = Z_0\frac{\partial}{\partial t} + Z_1\frac{\partial}{\partial x} + \left(Z_0\left(g + \frac{\dot{x}^2}{g}\right) - Z_1\frac{\dot{x}}{g}\right)\frac{\partial}{\partial y} + Z_2\frac{\partial}{\partial \dot{x}}. \tag{5.42}$$

Note that also in this case the canonical distribution is not projectable onto $\mathbb{R} \times \mathbb{R}^2$, so that variations of sections in Q_+ induced by Chetaev vector fields are not associated with variations induced by some vector fields in $\mathbb{R} \times \mathbb{R}^2$.

Writing the constrained first variation formula

$$\begin{aligned}\int_\Omega J^1\gamma^*\mathcal{L}_Z &\left(2c\,\mathrm{d}t - m\dot{x}\,\mathrm{d}x - m\sqrt{\frac{2c}{m} + 2Gx - \dot{x}^2}\,\mathrm{d}y\right)\\ &= \int_\Omega J^1\gamma^* i_Z d\left(2c\,\mathrm{d}t - m\dot{x}\,\mathrm{d}x - m\sqrt{\frac{2c}{m} + 2Gx - \dot{x}^2}\,\mathrm{d}y\right)\\ &\quad - \int_{\partial\Omega}(2mGxZ_0)\circ\gamma,\end{aligned} \tag{5.43}$$

and taking vertical Chetaev vector fields, we can easily obtain the constrained Euler–Lagrange equation:

$$\begin{aligned}\bar{h}i_Z\,\mathrm{d}&\left(2c\,\mathrm{d}t - m\dot{x}\,\mathrm{d}x - m\sqrt{\frac{2c}{m} + 2Gx - \dot{x}^2}\,\mathrm{d}y\right)\\ &= \bar{h}i_Z\left(-m\,\mathrm{d}\dot{x}\wedge\mathrm{d}x - \frac{mG}{g}\,\mathrm{d}x\wedge\mathrm{d}y + \frac{m\dot{x}}{g}\,\mathrm{d}\dot{x}\wedge\mathrm{d}y\right)\\ &= \left(m\ddot{x} - mG - \frac{mG}{g^2}\dot{x}^2 + \frac{m\dot{x}^2}{g^2}\ddot{x}\right)Z_1\,\mathrm{d}t = 0.\end{aligned} \tag{5.44}$$

Since the above relation holds for arbitrary Z_1, we finally obtain equation (5.37).

Note that the constrained system in this example can be described either by the constraint Lepage form $\bar{\rho}$ (5.39), or by *two* 'Lagrangian functions'

$$\overline{L} = -2mGx, \qquad L_1 = -m\sqrt{\frac{2c}{m} + 2Gx - \dot{x}^2}. \tag{5.45}$$

Since $\partial\overline{L}/\partial\dot{x} = 0$, we obtain in this case the constraint Lagrangian 1-form equal to $\bar{\rho}$, i.e.

$$\lambda_c = -2mGx\,\mathrm{d}t - \left(m\sqrt{\frac{2c}{m} + 2Gx - \dot{x}^2}\right)\varphi. \tag{5.46}$$

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References

[1] Binz E, de León M, de Diego D M and Socolescu D 2002 Nonholonomic constraints in classical field theories *Rep. Math. Phys.* **49** 151–66
[2] Cardin F and Favretti M 1996 On nonholonomic and vakonomic dynamics of mechanical systems with nonintegrable constraints *J. Geom. Phys.* **18** 295–325
[3] Cariñena J F and Rañada M F 1993 Lagrangian systems with constraints: a geometric approach to the method of Lagrange multipliers *J. Phys. A: Math. Gen.* **26** 1335–51
[4] Chetaev N G 1932-33 On the Gauss principle *Izv. Kazan. Fiz.-Mat. Obsc.* **6** 323–6 (in Russian)
[5] Czudková L, Janová J and Musilová J 2005 Non-holonomic mechanical systems and variational principle *Differential Geometry and its Applications: Proc. Conf. DGA (2004, Prague)* ed J Bures , O Kowalski, D Krupka and J Slovák (Prague: Matfyzpress) pp 571–9
[6] de León M and de Diego D M 1996 On the geometry of non-holonomic Lagrangian systems *J. Math. Phys.* **37** 3389–414
[7] de León M, Marrero J C and de Diego D M 1997 Non-holonomic Lagrangian systems in jet manifolds *J. Phys. A: Math. Gen.* **30** 1167–90
[8] Giachetta G 1992 Jet methods in nonholonomic mechanics *J. Math. Phys.* **33** 1652–65
[9] Grabowska K and Grabowski J 2008 Variational calculus with constraints on general algebroids *J. Phys. A: Math. Theor.* **41** 175204
[10] Koon W S and Marsden J E 1997 The Hamiltonian and Lagrangian approaches to the dynamics of nonholonomic systems *Rep. Math. Phys.* **40** 21–62
[11] Krupka D 1973 Some geometric aspects of variational problems in fibred manifolds *Folia Fac. Sci. Nat. UJEP Brunensis* **14** 1–65
Krupka D 2001 Electronic transcription arXiv:math-ph/0110005
[12] Krupka D 2008 Global variational theory in fibred spaces *Handbook of Global Analysis* ed D Krupka and D Saunders (Amsterdam: Elsevier) pp 773–836
[13] Krupková O 1997 *The Geometry of Ordinary Variational Equations (Lecture Notes in Mathematics* vol 1678*)* (Berlin: Springer)
[14] Krupková O 1997 Mechanical systems with nonholonomic constraints *J. Math. Phys.* **38** 5098–126
[15] Krupková O 2002 Recent results in the geometry of constrained systems *Rep. Math. Phys.* **49** 269–78
[16] Krupková O 2006 Partial differential equations with differential constraints *J. Diff. Eqn.* **220** 354–95
[17] Krupková O and Musilová J 2001 The relativistic particle as a mechanical systems with non-holonomic constraints *J. Phys. A: Math. Gen.* **34** 3859–75
[18] Krupková O and Musilová J 2005 Non-holonomic variational systems *Rep. Math. Phys.* **55** 211–20
[19] Krupková O and Volný P 2005 Euler–Lagrange and Hamilton equations for non-holonomic systems in field theory *J. Phys. A: Math. Gen.* **38** 8715–874
[20] Krupková O and Volný P 2006 Differential equations with constraints in jet bundles: Lagrangian and Hamiltonian systems *Lobachevskii J. Math.* **23** 95–150
[21] Krupková O, Volná J and Volný P 2008 Constrained Lepage forms *Differential Geometry and its Applications: Proc. Conf. DGA (2007, Olomouc)* ed O Kowalski, D Krupka, O Krupková and J Slovák (Singapore: World Scientific) pp 627–33
[22] Li S M and Berakdar J 2008 A Lagrangian formalism based on the velocity-determined virtual displacements for systems with nonholonomic constraints *Int. J. Theor. Phys.* **47** 732–40
[23] Massa E and Pagani E 1997 A new look at classical mechanics of constrained systems *Ann. Inst. Henri Poincaré* **66** 1–36
[24] Sarlet W 1996 A direct geometrical construction of the dynamics of non-holonomic Lagrangian systems *Extracta Mathematicae* **11** 202–12
[25] Sarlet W, Cantrijn F and Saunders D J 1995 A geometrical framework for the study of non-holonomic Lagrangian systems *J. Phys. A: Math. Gen.* **28** 3253–68
[26] Saunders D J, Sarlet W and Cantrijn F 1996 A geometrical framework for the study of non-holonomic Lagrangian systems II. *J. Phys. A: Math. Gen.* **29** 4265–74

[27] Swaczyna M 2004 On the nonholonomic variational principle *Global Analysis and Applied Mathematics: Proc. International Workshop on Global Analysis (Ankara, Turkey, 2004) (AIP Conf. Proc. (Melville, New York)* pp 297–306

[28] Swaczyna M and Tichá M 2004 *Examples of nonholonomic mechanical systems Preprint Series in Global Analysis and Applications* No. 10 *(Department of Algebra and Geometry, Palacky University, Olomouc)* pp 1–24

[29] Vankerschaver J 2007 A class of nonholonomic kinematic constraints in elasticity *J. Phys. A: Math. Theor.* **40** 3889

[30] Vankerschaver J, Cantrijn F, de León M and Martín de Diego D 2005 Geometric aspects of nonholonomic field theories *Rep. Math. Phys.* **56** 387–411